Universal Codes for Parallel Gaussian Channels

by

Maryam Modir Shanechi
B.A.Sc., Engineering Science (2004)
University of Toronto

Submitted to the Department of Electrical Engineering and Computer Science
in partial fulfillment of the requirements for the degree of

Masters of Science in Electrical Engineering and Computer Science

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2006

Author ....
Department of Electrical Engineering and Computer Science
May 12, 2006

Certified by ....
Gregory W. Wornell
Professor, Department of Electrical Engineering and Computer Science
Thesis Supervisor

Certified by ....
Dr. Uri Erez
Department of Electrical Engineering - Systems, Tel Aviv University
Thesis Supervisor

Accepted by ....
Arthur C. Smith
Chairman, Department Committee on Graduate Students

Universal Codes for Parallel Gaussian Channels

by

Maryam Modir Shanechi

Submitted to the Department of Electrical Engineering and Computer Science
on May 12, 2006, in partial fulfillment of the
requirements for the degree of
Masters of Science in Electrical Engineering and Computer Science

Abstract

In this thesis we study the design of universal codes for parallel Gaussian channels with 2 sub-channels present. We study the universality both in terms of the uncertainty in the relative quality of the two sub-channels for a fixed maximum rate, C^*, and in terms of the uncertainty of the achievable maximum rate. In our architecture, we will convert the parallel Gaussian channel into a set of scalar Gaussian channels and use good base codes designed for the corresponding scalar channel in the coding schemes.

In Chapter 2, a universal layered code with deterministic dithers is developed. The code is repeated across the two sub-channels with possibly different dithers. Symbols in each of the layer codewords can be combined using unitary transformations of dimension, m. A minimum mean squared error (MMSE) receiver combined with successive cancellation is used for decoding. We show that increasing m does not improve the efficiency. The efficiency increases by adding more layers up to a certain number and after that it saturates. We find an expression for this saturation efficiency. We show that partial CSIT improves the efficiency significantly. At the end we compare the performance of maximal ratio combining (MRC) and MMSE receivers and show that they are close in the coding scheme with no CSIT.

In Chapter 3, we design an alternative universal code and extend it to be rateless. This is a sub-block structured code symmetric with respect to all layers that gets repeated across the two sub-channels and in time using i.i.d. Bernoulli (1/2) dithers. The decoder uses an MRC receiver combined with successive cancellation. We prove that in the limit of large L when L is increased exponentially with C^*, the code is capacity achieving. We perform efficiency analyses when L is scaled linearly with C^* and derive upper and lower bounds on the efficiency. We also show that the scheme has high efficiencies for practical ranges of C^* using a low-rate good base code. We discuss the unknown time-varying behavior of the scheme and at the end briefly discuss the use of faster than Nyquist signaling to enable the scheme to have a high efficiency for higher C^* values.

Thesis Supervisor: Gregory W. Wornell
Title: Professor, Department of Electrical Engineering and Computer Science

Thesis Supervisor: Dr. Uri Erez
Title: Department of Electrical Engineering - Systems, Tel Aviv University

Acknowledgments

This work could not have been possible if not for the help and support from all my colleagues and friends. I am particularly thankful to Gregory Wornell and Uri Erez for introducing me to this problem and giving me directions especially in the early stages of the project. I am greatly thankful to my office mate, Ying-zong Huang for all his wonderful discussions and help throughout the course of my research and for proof reading my thesis. I would also like to thank Charles Swannack for helping me with all my network problems and latex questions and for his great research discussions. I am also thankful to Ashish Khisti for his valuable research discussions and brainstorming. I would also like to thank all the members of the SIA group who made my experience so enjoyable, especially Tricia Mulcahy for all her kindness and care. I am also very grateful to Pablo Parrilo for all his help with my research and optimization questions.

I am greatly indebted to all my friends for their loving support and care. I would especially like to thank my roommate and best friend, Vivienne Sze, who was always there for me whenever I needed her, in moments of happiness, sadness or pure panic. I would also like to thank my dear friend Bahareh for all her love and care and especially the cookies she brought me during my qualification exam. I am grateful to all my friends for making my experience extremely pleasant at MIT. Rumi for her warm laugh. Behnam for keeping my mind off the research when I needed it. Fardad for introducing me to Tealuxe. Joyce, Denis, Ardavan, Babak, Farinaz, and Elnaz for their great companionship.

Finally I would like to thank all members of my family, mom, dad, Mercedeh and Amir for being there for me every step of the way. I could not have dreamt of getting here without you and all your sacrifices. Thank you from the bottom of my heart.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Many communication channels can be modeled and analyzed as parallel Gaussian channels with a total power constraint. For example in wireless communication, a multiple input multiple output (MIMO) channel can be transformed and decomposed into a set of parallel independent scalar Gaussian channels using the singular value decomposition of the channel matrix H [9]. Other examples of channels that can be modeled as parallel Gaussian channels are time-invariant frequency-selective channels and time-varying fading channels. In the fading channels each sub-channel corresponds to a fading state and in the frequency selective channels, each sub-channel corresponds to an independent sub-carrier [9].

Achieving capacity in parallel additive white Gaussian noise (AWGN) channels with a total power constraint and when the sub-channel noise levels are known, is well understood. Capacity in these channels can be achieved by an optimal power allocation over the sub-channels. The optimal power for each sub-channel is found by a waterfilling allocation [1]. However, when noise levels are unknown, the best that can be done is to use an equal power allocation for each sub-channel while still satisfying the total power constraint. An example of this situation is the fast fading channel with channel side information (CSI) available only at the receiver (CSIR). Since the transmitter does not know the CSI, no waterfilling can be done across the independent fading states (sub-channels) [9].

Design of practical universal codes for parallel Gaussian channels with unknown channel state at the transmitter and with CSIR is of great interest because of their great modeling power in many practical communication channels. In [8], [9] permutation codes have been designed as approximately universal codes for parallel Gaussian channels. In these works, a universal design criterion is derived. This criterion is based on the worst-case realization of the channel that is not in outage. It is shown that the universal design criterion at high SNR reduces to choosing codewords that maximize the pairwise product distance. This means that at high SNR, the universal design criterion is the same as the product distance criterion which is also the design criterion for i.i.d. Rayleigh parallel fading channels. It is shown that random permutation codes are approximately universal for the parallel channel with high probability. These codes are space-only unit block length codes that make use of a different permutation of the same QAM constellation for transmission over different parallel sub-channels.

This thesis is motivated by the need to design low-complexity universal codes for parallel Gaussian channels. In our design architecture, we will convert the parallel Gaussian channel into a set of scalar Gaussian channels and hence use low-complexity 'good' base codes for the corresponding scalar channel to communicate. Basically in our design, the code effectively sees a scalar Gaussian channel. Hence our architecture is different from that of [8], [9] which use codes with a product distance criterion at high SNR. We will show that in our design in Chapter 2, one can use a good AWGN base code to communicate. The base code design in Chapter 3 is more complicated and one needs codes designed for time-varying Gaussian scalar channels.

The problem of universal coding over parallel channels bears interesting similarities to the problem of rateless coding in scalar channels. Rateless codes are infinite length codes whose prefixes are also good codes. These codes are useful for situations when the channel conditions are unknown. They enable the transmitter to send the same code for all channel qualities. Depending on the specific channel quality realized, the receiver in turn collects as much of the codeword as it needs to accumulate enough mutual information and decode. Hence, just as in the parallel channel problem where there is uncertainty in the quality realization of the sub-channels, in the rateless coding problem there is uncertainty about the quality of the scalar channel.

Low-complexity rateless codes for erasure channels, known as Raptor codes, have been designed in [6] which are an extension of LT codes [5]. For AWGN channels, low-complexity capacity approaching rateless codes have been developed in [3] for a scalar Gaussian channel that employ a good AWGN binary base code of low rate. Layering along with non-uniform time-varying power allocation, multiplicative i.i.d. equiprobable ± 1 (Bernoulli $(1/2)$) dithering, and repetition are strategies used to obtain a block-structured rateless code. Dithering makes the repetition blocks of the base codeword uncorrelated to reduce the mutual information penalty due to temporal repetition. The decoder structure uses a simple maximal ratio combining (MRC) receiver along with successive cancellation. The dithering will then allow the MRC receiver to approach capacity at a large number of layers. The power allocations are designed such that all layers contribute the same amount of rate. The number of layers is scaled linearly with capacity. It is shown that using a good standard binary code of sufficiently low rate, any desired fraction of capacity can be achieved over the scalar Gaussian channel. For example using an AWGN codebook of rate 1/7 per layer is enough to get to 90% efficiency.

To see the similarity between the rateless coding problem for the Gaussian scalar channel and the universal coding problem for the parallel Gaussian channel, note that in the rateless problem, the Gaussian channel quality is unknown and the rateless code works close to capacity for any quality realization of the channel. In the parallel channel problem, the sub-channel qualities are unknown and the objective is to get close to capacity for any overall channel realization. Our goal is to extend the ideas behind the block-structured design in [3] to universal coding for parallel Gaussian channels.

We can model the uncertainty in the parallel channel by breaking it into two parts:

(1) The uncertainty in the relative quality of the two sub-channels: Having a fixed known overall capacity, C^*, the sub-channels can take on any realization pair $(\mathrm{SNR}_1, \mathrm{SNR}_2)$, as long as the pair satisfies:

$$\log(1+\mathrm{SNR}_1) + \log(1+\mathrm{SNR}_2) = C^* \tag{1.1}$$

(2) The uncertainty in the overall maximum achievable rate: The overall capacity, C^*, can change.

To model both types of uncertainty, we can introduce a positive integer parameter, r. Assuming that the maximum possible capacity is C^*, the overall channel capacity can take any value, $C(r) = C^*/r$. Hence the channel realizations can be modeled as all pairs of the form, $(\mathrm{SNR}_1(r), \mathrm{SNR}_2(r))$, that satisfy:

$$\log(1+\mathrm{SNR}_1(r)) + \log(1+\mathrm{SNR}_2(r)) = \frac{C^*}{r}$$

where r can be any positive integer. Figure 1-1 shows the possible pairs for $C^* = 1$ bit and for different values of r.

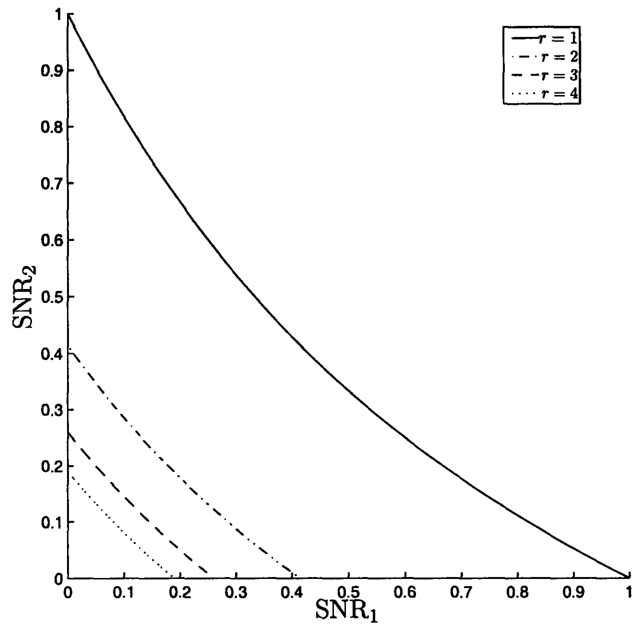


Figure 1-1. Channel realizations when $C^* = 1$ bit. Different curves correspond to different maximum achievable rates, C^*/r, in the channel. Every single curve corresponds to the possible sub-channel realizations for a fixed maximum rate.

To address the problem of the uncertainty of the first type, we must design universal codes that can achieve rates close to C^* for any pair realization. To address the problem of the uncertainty of the second type, we must complement the universal code to be rateless as well.

The rateless design is similar to that done in [3]. However, we have more requirements since in that work, the realization pair uncertainty was not present.

The subsequent chapters of this thesis are organized as follows: In Chapter 2, we design universal codes that address the first type of uncertainty. Even though this design can be extended to include the rateless case as well, this will not be the focus of the second chapter to keep the exposition compact. The motivation for addressing only the first type of uncertainty comes from the outage definition in slow fading channels where the outage event here for a given maximum rate, C^*, is defined as all the SNR pair realizations for which the sum in (1.1) is less than C^*. Thus, we require the code to be capacity achieving as long as there is no outage, i.e., the SNR pair does not belong to the outage event. In this design, we use layering and non-uniform power allocation across the layers. However, instead of using random Bernoulli (1/2) dithers, we will use deterministic dithering. At the receiver a minimum mean squared error (MMSE) receiver is used along with successive cancellation. The design problem optimizes over the power allocations and the deterministic dithers to achieve the best possible efficiency. The roles of the number of layers, dither dimension, partial channel information, and decoder structure on the efficiency of the scheme are studied. Also the robustness of the scheme to channel knowledge is investigated.

In Chapter 3, we consider both kinds of uncertainty and design an alternative universal code for the parallel Gaussian channel and extend the design to be rateless. In this chapter we develop a sub-block structured code using layering, staggering and overlapping of codewords, random Bernoulli (1/2) dithering, and repeating (if rateless). The decoder uses the simple MRC receiver along with successive cancellation. We analyze the efficiency of the scheme and show that in order to get to capacity for this scheme, number of layers should increase exponentially with capacity. However, we show that for all practical C^* values, scaling the number of layers linearly with C^* will still result in reasonably high efficiencies using a low rate base code. We also briefly study the use of faster than Nyquist (FTN) signaling to be able to achieve a high efficiency at higher values of C^*. Finally we discuss the unknown time-varying nature of this coding scheme and its implications on the performance of base codes especifically that of good low-rate AWGN binary codes at low SNR.

In Chapter 4, we summarize the conclusions of the analyses and give some future research directions.

Chapter 2

Layered Code Design with Deterministic Dither

In this chapter, we focus on designing a layered universal code for the parallel Gaussian channel with deterministic dithers. Even though we can generalize the design to the rateless case, to keep the exposition compact, here we focus on the non-rateless situation. In other words, we fix an overall maximum rate, C^*, for the parallel channel and design a code that can achieve a rate close to this maximum for any realization of the channel. As mentioned in Chapter 1, the motivation for addressing only the first type of uncertainty comes from the outage definition in slow fading channels. The outage event here for a given maximum rate, C^*, is defined as all the SNR pair realizations for which the sum (cf. (1.1)):

$$\log(1+\mathrm{SNR}_1)+\log(1+\mathrm{SNR}_2) \tag{2.1}$$

is less than C^*. Thus, we require the code to be capacity achieving as long as there is no outage, i.e., the SNR pair does not belong to the outage event.

We can characterize the channel as consisting of the SNR pairs where the sum in 2.1 is equal to C^*. For simplicity (when we later find the power allocations), here we normalize the total power in each sub-channel to be 1. Thus, we can instead characterize the channel by the noise pairs, (N_1, N_2), such that,

$$\log(1+\frac{1}{N_1})+\log(1+\frac{1}{N_2})=C^*$$

We can therefore parameterize the noises in terms of a single parameter, t, as:

$$N_1(t) = \frac{1}{e^{\frac{C^*}{2}-t}-1} \tag{2.2}$$

$$N_2(t) = \frac{1}{e^{\frac{C^*}{2}+t}-1} \tag{2.3}$$

$$|t| \leq \frac{C^*}{2}$$

The main tools used in the code design are:

- layering,
- deterministic optimal dithering, and
- grouping the symbols and combining these groups using unitary transformations.

.

By grouping symbols, we introduce additional degrees of freedom in the code design, i.e., change the dither dimension in the code. Restricting the transformation to be unitary preserves the AWGN property of the channel after the grouping by keeping the interference vector Gaussian. The decoder structure used consists of an MMSE receiver in combination with successive cancellation.

We start the chapter by developing a general code structure and formulating an optimization problem for code design. We then look at the structure of the solutions to the optimal code design problem and derive upper bounds on the efficiency of the scheme as a function of the number of layers. We then solve the optimization code design problem and obtain numerical solutions for some specific choices of C^* values and show that the upper bounds can be made tight up to a certain number of layers. Finally we derive an ultimate upper bound on the efficiency of the scheme for an arbitrary maximum rate and show that it can be made tight at a large enough number of layers. Guided by these results, we make conclusions on the effect of different parameters such as the dither dimension or the number of layers on the efficiency performance of the coding scheme.

We next examine possible improvements to the scheme by studying the effect of partial channel side information at the transmitter (CSIT) on the efficiency performance and show that even a single bit of information can make a significant improvement in performance. We also examine the robustness of the scheme to unreliable channel knowledge.

At the end, we study the effect of the decoder structure on the efficiency performance in the case of code design without partial channel information at the transmitter. More specifically, we redesign an optimal code assuming the simple MRC receiver at the decoder instead of the more advanced MMSE receiver and compare its performance to the later.

2.1 Code Structure and Achievable Rates

Here we develop the general code structure and examine the resulting channel model. From that we find the achievable rate formulation in the coding scheme for different layers as a function of the code parameters.

Encoding Structure

As mentioned earlier, layering, dithering, and grouping are the main tools used in this coding scheme. The coding scheme can have different number of layers and can group and combine different number of symbols within the layers using unitary matrices. In general we have L streams of codewords, where L is the number of layers. We combine every m symbols (m is index in time) in each of these L streams using unitary matrices in the overall dither matrix and send the combined result over the two sub-channels. To see this, let's denote a group of m symbols in the lth stream by[1]:

$$\mathbf{c}_l^t = \begin{bmatrix} \mathbf{c}_l(1) & \mathbf{c}_l(2) & \cdots & \mathbf{c}_l(m) \end{bmatrix}$$

Figure 2-1 shows the L streams of codewords with a base code block length of n. We split the n symbols in a codeword into groups of m symbols and apply the unitary transformations on the symbols in each group.

Denoting the corresponding symbols sent over sub-channel 1 and sub-channel 2 by:

$$\mathbf{x}_1^t = \begin{bmatrix} \mathbf{x}_1(1) & \mathbf{x}_1(2) & \cdots & \mathbf{x}_1(m) \end{bmatrix}$$

and

$$\mathbf{x}_2^t = \begin{bmatrix} \mathbf{x}_2(1) & \mathbf{x}_2(2) & \cdots & \mathbf{x}_2(m) \end{bmatrix}$$

[1]The superscript t denotes the transpose of the vector.

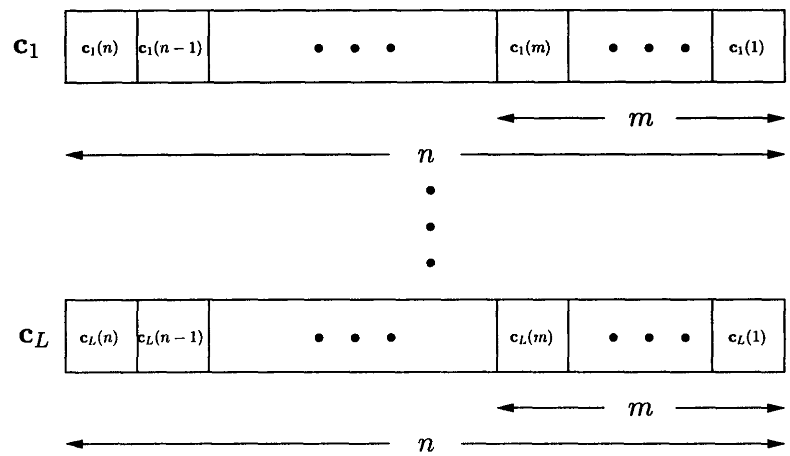


Figure 2-1. L streams of codewords for the L layers. Each codeword has a block length of n. The symbols within each codeword are split in groups of m symbols and the unitary transformations are applied on the members of each group. The resulting groups after the transformations are added together and sent over the two sub-channels.

respectively, our encoding scheme has the following structure:

$$\begin{aligned}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} &= \begin{bmatrix} \sqrt{a_1}\mathbf{U}_1 & \sqrt{a_2}\mathbf{U}_2 & \cdots & \sqrt{1-\sum_{i=1}^{L-1} a_i}\mathbf{U}_L \\ \sqrt{b_1}\mathbf{U}_{L+1} & \sqrt{b_2}\mathbf{U}_{L+2} & \cdots & \sqrt{1-\sum_{i=1}^{L-1} b_i}\mathbf{U}_{2L} \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_L \end{bmatrix} \\ &= \mathbf{U} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_L \end{bmatrix}\end{aligned} \tag{2.4}$$

where $\mathbf{U}_j$'s are unitary matrices of size $m \times m$ and a_i's and b_i's are the power allocations for layer i and in sub-channels 1 and 2 respectively. Here $\mathbf{U}$ denotes the overall dither matrix. As mentioned before, the reason to use unitary matrices is to maintain the AWGN property of the channel. We can now use good base codes for an AWGN channel to construct the layer codewords.

Channel Model

Using the encoding structure developed in (2.4), the channel input-output relationship can be written as:

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \mathbf{U}_1 & \alpha_2 \mathbf{U}_2 & \cdots & \alpha_L \mathbf{U}_L \\ \beta_1 \mathbf{U}_{L+1} & \beta_2 \mathbf{U}_{L+2} & \cdots & \beta_L \mathbf{U}_{2L} \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_L \end{bmatrix} + \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} \tag{2.5}$$

where,

$$\begin{cases} \alpha_i = \sqrt{\frac{a_i}{N_1}} & i = 1, \cdots, L-1 \\ \alpha_L = \sqrt{\frac{1-\sum_{i=1}^{L-1} a_i}{N_1}} \end{cases}$$

$$\begin{cases} \beta_i = \sqrt{\frac{b_i}{N_2}} & i = 1, \cdots, L-1 \\ \beta_L = \sqrt{\frac{1-\sum_{i=1}^{L-1} b_i}{N_2}} \end{cases}$$

and

$$\mathbf{w} = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} \sim \mathcal{CN}(0, \mathbf{I})$$

Decoding Structure and Achievable Rates

Now to find the achievable rate in each layer of the coding scheme we should look at the decoder structure. At the decoder, we use an MMSE receiver along with successive cancellation to decode the layers. The MMSE receiver makes use of the knowledge of the deterministic dither matrices. We know that the MMSE receiver for Gaussian channels provides a sufficient statistic and is information lossless [9]. Hence using the MMSE receiver along with successive cancellation, the achievable rate in the kth layer is given by[2]:

$$R_k = \frac{1}{m} \log \left[\det \left(\mathbf{I} + \begin{bmatrix} \alpha_1 \mathbf{U}_1 & \cdots & \alpha_k \mathbf{U}_k \\ \beta_1 \mathbf{U}_{L+1} & \cdots & \beta_k \mathbf{U}_{L+k} \end{bmatrix} \begin{bmatrix} \alpha_1 \mathbf{U}_1 & \cdots & \alpha_k \mathbf{U}_k \\ \beta_1 \mathbf{U}_{L+1} & \cdots & \beta_k \mathbf{U}_{L+k} \end{bmatrix}^H \right) \right] - R_{k-1} \tag{2.6}$$

where m here, as previously defined, is the number of symbols combined using unitary transformations during encoding.

We can also look at the channel model in (2.5) as a MIMO channel with 2 receive antennas, L transmit antennas with equal power allocation ($\mathbf{c}_l$'s are iid $\mathcal{CN}(0, \mathbf{I})$), and channel

[2]Here the superscript H denotes the conjugate transpose of the matrix.

matrix $\mathbf{U}$. For such a MIMO channel with channel state information at the receiver, the MMSE receiver combined with successive cancellation and equal power allocation achieves capacity [9]. This again leads us to (2.6) for the achievable rates.

2.2 Problem Formulation for Optimal Code Design

Having introduced the coding structure, we will now formulate the code design problem to find the optimal values of different parameters. In the coding structure, we have control over the number of degrees of freedom available and can make it arbitrarily large. Each of the unitary matrices, $\mathbf{U}_j$, has m^2 real degrees of freedom. We also have $2L-2$ power allocation degrees of freedom. The goal should be to optimize over these degrees of freedom to achieve an overall rate as close to capacity as possible for any realization.

To formulate the design optimization problem, we should first establish the method by which we allocate rates to different layers. In our coding scheme, we will use a common single base codebook of a certain rate for all the layers. This will eliminate the need to use different codebooks of different rates for each layer and simplify the coding scheme. Let's denote this common rate by R^*. Note that here the base codebook is a good codebook for the AWGN channel of rate R^*.

Using a common rate for all the layers means that we have to design for their minimum rate. Let's denote the minimum rate among all the layers for a specific dither matrix, $\mathbf{U}$, and at a certain channel realization, t, by $R(t, \mathbf{U})$. Here t is the parameter which identifies the channel noise pair realization in (2.2) and (2.3). For the best design, we should find the optimal values of the available degrees of freedom that maximize the rate of the base codebook achievable in all layers at all channel realizations. Hence, for a fixed m and L, the code design problem reduces to the following optimization problem:

$$R^* = \max_{\mathbf{U}} \min_{t} R(t, \mathbf{U}) = \max_{\mathbf{U}} \min_{t} \min_{l} R_l(t, \mathbf{U}) \tag{2.7}$$

s.t.

$$\begin{aligned}
&\mathbf{U}_j \mathbf{U}_j^H = \mathbf{I} && j = 1, \cdots, 2L \\
&\sum_{i=1}^{k} a_i \leq 1 && \forall k = 1, \cdots, L-1 \\
&\sum_{i=1}^{k} b_i \leq 1 && \forall k = 1, \cdots, L-1
\end{aligned}$$

where from (2.4),

$$\mathbf{U} = \begin{bmatrix} \sqrt{a_1}\mathbf{U}_1 & \sqrt{a_2}\mathbf{U}_2 & \cdots & \sqrt{1 - \sum_{i=1}^{L-1} a_i}\mathbf{U}_L \\ \sqrt{b_1}\mathbf{U}_{L+1} & \sqrt{b_2}\mathbf{U}_{L+2} & \cdots & \sqrt{1 - \sum_{i=1}^{L-1} b_i}\mathbf{U}_{2L} \end{bmatrix}$$

and the rates, $R_l(t, \mathbf{U})$, are calculated according to (2.6).

We can now define the efficiency of the scheme as the ratio of the total rate we can send through the channel using our coding scheme to capacity, i.e., $\eta = R^* L / C^*$. Using this definition, the above optimization problem aims to maximize the efficiency of the scheme.

Note that introducing additional degrees of freedom makes the encoding and decoding more complex. Hence we should only use as many degrees of freedom as necessary to achieve a high efficiency. In the later sections of this chapter, we derive an upper bound on the efficiency and show that once we can make this upper bound tight using a certain number of degrees of freedom, there will be no need to go to higher degrees.

■ 2.3 Upper-bounds on Efficiency

Before solving the optimization design problem, it will be beneficial to examine the structure of the code and find some upper bound on the efficiency that would give us insight into the problem. Since we have the option of increasing the number of degrees of freedom in the problem, this upper bound will later help determine whether using more degrees of freedom achieves higher efficiencies. In other words, if using a certain number of degrees of freedom we can make the upper bound tight, there will be no need to introduce additional degrees of freedom.

To derive this upper bound, we revisit the channel model given in (2.5) and examine its special structure. As we will show in what follows, the achievable rate in the first layer takes a simple form which will help us derive the upper bound on the efficiency. Hence we will first examine the behavior of the achievable rate in the first layer and later use its properties to find the upper bound.

2.3.1 Achievable Rate in the First Layer

Let's calculate the rate in layer 1, namely, R_1:

$$\begin{aligned} R_1 &= \frac{1}{m}\log\left[\det\left(\mathbf{I}+\begin{bmatrix}\alpha_1\mathbf{U}_1\\ \beta_1\mathbf{U}_{L+1}\end{bmatrix}\begin{bmatrix}\alpha_1\mathbf{U}_1\\ \beta_1\mathbf{U}_{L+1}\end{bmatrix}^H\right)\right] \\ &= \frac{1}{m}\log\det\left(\mathbf{I}+\begin{bmatrix}\alpha_1^2\mathbf{I} & \alpha_1\beta_1\mathbf{U}_1\mathbf{U}_{L+1}^H\\ \alpha_1\beta_1\mathbf{U}_{L+1}\mathbf{U}_1^H & \beta_1^2\mathbf{I}\end{bmatrix}\right) \\ &= \frac{1}{m}\log\det\begin{bmatrix}(1+\alpha_1^2)\mathbf{I} & \mathbf{A}\\ \mathbf{A}^H & (1+\beta_1^2)\mathbf{I}\end{bmatrix} \end{aligned} \tag{2.8}$$

where $\mathbf{A} = \alpha_1\beta_1\mathbf{U}_1\mathbf{U}_{L+1}^H$ and as defined before $\alpha_i = \sqrt{a_i/N_1}$, $\beta_i = \sqrt{b_i/N_2}$ and a_i and b_i are the powers in layer i of the two sub-channels. We observe that $\mathbf{A}$ is a scaled unitary matrix. The above matrix has a special structure. It is hermitian and the off diagonal blocks are unitary. From [7], if $\mathbf{M} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}$ and $\mathbf{CD} = \mathbf{DC}$ then:

$$\det(\mathbf{M}) = \det(\mathbf{AD} - \mathbf{BC})$$

This condition is satisfied in the matrix involved in R_1 calculation. Hence we have:

$$\begin{aligned} \det\begin{bmatrix}(1+\alpha_1^2)\mathbf{I} & \mathbf{A}\\ \mathbf{A}^H & (1+\beta_1^2)\mathbf{I}\end{bmatrix} &= \det\left((1+\alpha_1^2+\beta_1^2)\mathbf{I}_{m\times m}\right) \\ &= (1+\alpha_1^2+\beta_1^2)^m \end{aligned}$$

Now substituting into (2.8), R_1 is given by:

$$R_1 = \frac{1}{m}\log(1+\alpha_1^2+\beta_1^2)^m$$
$$= \log(1+\alpha_1^2+\beta_1^2) \tag{2.9}$$
$$= \log(1+\frac{P_1(1)}{N_1}+\frac{P_1(2)}{N_2}) \tag{2.10}$$

Here we define $P_1(1)$ and $P_1(2)$ as the first layer powers in sub-channels 1 and 2 respectively. Therefore $\alpha_1^2 = P_1(1)/N_1$ and $\beta_1^2 = P_1(2)/N_2$ where we can again parameterize the different noise pair realizations according to (2.2) and (2.3).

Note that this result holds for any L and m and hence is independent of the number of layers or the number of symbols grouped together. Hence, the rate in the first layer is only a function of the power allocation of the two sub-channels in the first layer and the noise variances or equivalently the channel realization parameterized by t, and is independent of the other degrees of freedom in the dither matrix. Therefore, from now on, we will denote this rate as either R_1 or $R_1(t)$ to make this dependence explicit.

2.3.2 Upper Bound

Here we will continue to derive an upper bound on the efficiency, by examining the properties of the optimization problem and using the first layer rate derivation in (2.10). As mentioned in the previous sections, in the coding scheme, we use a common base codebook for all the layers and hence we should design for the minimum achievable rate among them and in the worst channel realization. Therefore, for the purpose of finding the maximum efficiency, it is important to find the behavior of the minimum rate in layer 1 as a function of the channel realization. To see where the minimum of $R_1(t)$ happens we differentiate with respect to t:

$$\frac{\partial R_1}{\partial t} = 0 \Rightarrow t = \frac{1}{2}\log(\frac{P_1(1)}{P_1(2)}) \tag{2.11}$$

Here the problem is symmetric with respect to the power allocation for the two sub-channels and therefore $P_1(1) = P_1(2) = P_1$. Hence, layer 1 obtains its minimum rate when $t = 0$ or equivalently $N_1 = N_2 = 1/(e^{\frac{C^*}{2}} - 1)$. $R_1(t)$ is also a convex function of t, meaning it is maximum at $t = \pm C^*/2$, i.e., at either $(N_1 = 1/(e^{C^*} - 1), N_2 = \infty)$ or $(N_1 = \infty, N_2 =$

$1/(e^{C^*}-1))$. Summarizing these observations we have:

$$
\begin{aligned}
\arg\min_t R_1 &= 0 \\
\arg\max_t R_1 &= \pm\frac{C^*}{2}
\end{aligned}
$$

Now substituting into (2.10) for the first layer rate we get:

$$
\begin{aligned}
&\max_t R_1(t) = R_1(\pm\frac{C^*}{2}) = \log(1+P_1(e^{C^*}-1)) \\
&\min_t R_1(t) = \log(1+2P_1(e^{\frac{C^*}{2}}-1))
\end{aligned}
$$

This holds for any m and for any L. Since in the coding scheme we select a rate which is the minimum of the rate of all the layers, at the optimal efficiency solution, the optimal $\mathbf{U}$ and power allocation must be such that the minimum rates in all the layers are equalized. This means that at the optimum point:

$$
\min_t R_i(t,\mathbf{U}) = \min_t R_j(t,\mathbf{U}) \qquad \forall i,j
$$

otherwise we can redistribute the powers to bring up the lower rates and increase the efficiency. Here we have made the dependence of the rate of all the other layers on both the dither matrix, $\mathbf{U}$, and the channel realization, t, explicit. From now on we will call a power allocation which will equalize these rates in combination with some $\mathbf{U}$ matrix (this may still not be the optimum), a valid power allocation. As a result, the efficiency for some valid power allocation, P_1, is given by:

$$
\eta = L\frac{\min_t R_1(t)}{C^*} = \frac{L\log(1+2P_1(e^{\frac{C^*}{2}}-1))}{C^*} \tag{2.12}
$$

To find an upper bound on this efficiency, i.e., find the best case valid power allocation with a corresponding $\mathbf{U}$, let's look at the channel realization with $t = \pm C^*/2$. At these realizations, the coding scheme is capacity achieving (since one channel is off and there is no repetition). Hence, the sum rate of all the other layers is given by:

$$
\sum_{i=2}^{L} R_i(t=\pm\frac{C^*}{2},\mathbf{U}) = C^* - R_1(\pm\frac{C^*}{2}) = C^* - \log(1+P_1(e^{C^*}-1)) \qquad \forall m \tag{2.13}
$$

This is the exact sum rate of all the other layers at this realization. We therefore conclude that the sum of minimum rates of these layers is at most equal to the sum rate at this capacity achieving realization for a given power allocation, P_1. Hence in the best possible case, the dither matrix and power allocations are such that this realization happens to have a sum rate for these layers which is equal to the sum of minimum rates for them.

As discussed before, the optimal $\mathbf{U}$ and power allocation will equalize the minimum rates in all the layers. Hence in the best case described above, the sum of minimum rates will happen at $t = \pm C^*/2$ realization and all the layers will have an equal share of this sum rate. In other words, all layers except layer 1 will have their minimum rate at the $t = \pm C^*/2$ realization and their rates will satisfy:

$$
\begin{aligned}
\min_t R_i(t, \mathbf{U}) &= R_i(t = \pm \frac{C^*}{2}, \mathbf{U}) \\
&= \frac{C^* - \log(1 + P_1(e^{C^*} - 1))}{L - 1} \\
&= \min_t R_1(t) \\
&= \log(1 + 2P_1(e^{\frac{C^*}{2}} - 1)) \qquad \forall i \geq 2
\end{aligned}
$$

where the rate of all the layers at this realization are given by:

$$
R_i(t = \pm \frac{C^*}{2}, \mathbf{U}) = \log\left(1 + \sum_{j=1}^{i} P_j(e^{C^*} - 1)\right) - \log\left(1 + \sum_{j=1}^{i-1} P_j(e^{C^*} - 1)\right) \forall i \geq 1
$$

irrespective of $\mathbf{U}$. Hence starting from layer 1 we can recursively solve for all the power allocations. Note that picking the above optimum layer power allocations does not guarantee the feasibility of this optimal condition. To get to these power allocations, we made a major assumption that all layers except 1 attain their minimum rate at $t = \pm C^*/2$. This is not necessarily the case unless the dither matrix satisfies certain conditions that make this assumption possible. We will discuss this point more later. But for now we can say that if the dither matrix can be selected appropriately such that the best case assumption holds, then the optimal power allocations are given by the recursive equations given above.

To make this clear, let's now look at an example with 2 layers. In this case, the rate in layer 2 satisfies:

$$\min_{t,\mathbf{U}} R_2(t,\mathbf{U}) \leq C^* - R_1(\pm\frac{C^*}{2}) = C^* - \log(1+P_1(e^{C^*}-1)) \qquad \forall m$$

and the best case happens when:

$$\min_{t,\mathbf{U}} R_2(t,\mathbf{U}) = C^* - R_1(\pm\frac{C^*}{2}) = C^* - \log(1+P_1(e^{C^*}-1)) \qquad \forall m$$

Hence the optimum power allocation for $L=2$ in this best situation is given by solving:

$$\min_{t,\mathbf{U}} R_2(t,\mathbf{U}) = \min_t R_1(t)$$
$$\Rightarrow C^* - \log(1+P_1(e^{C^*}-1)) = \log(1+2P_1(e^{\frac{C^*}{2}}-1))$$

Equating the two we get the following solution for the optimum power allocation for the two layers:

$$\begin{aligned} P_1^* &= -\frac{1}{4}\frac{e^{C^*/2}+3-\sqrt{9e^{C^*}-2e^{\frac{C^*}{2}}+1+8e^{\frac{3C^*}{2}}}}{e^{C^*}-1} \\ P_2^* &= 1-P_1^* \end{aligned}$$

and the resulting upper bound is:

$$\eta_{\text{up}} = \log(1+2P_1^*(e^{\frac{C^*}{2}}-1))/C^*$$

Figure 2-2 shows the upper bound as a function of the number of layers for $C^* = 4.33$ bits (3 nats). Note that this upper bound may not be tight at all L's. In order to be able to achieve it, we need certain conditions on the dither matrices. For example the overall dither matrix should allow a high enough total rate at all channel realizations such that the minimum of none of the layer rates goes below their value at $t = \pm C^*/2$. For example for

$L = 2$ and to get to the upper-bound, $\mathbf{U}$ must be such that:

$$\begin{aligned}
R_{\text{tot}}(t) &\geq R_1(t) + \left[C^* - \log(1 + P_1(e^{C^*} - 1))\right] \qquad \forall t \\
&= \log(1 + \frac{P_1}{N_1(t)} + \frac{P_1}{N_2(t)}) + [C^* - \log(1 + P_1(e^{C^*} - 1))] \qquad \forall t \\
&= C^* - \log \frac{1 + P_1(e^{C^*} - 1)}{1 + \frac{P_1}{N_1(t)} + \frac{P_1}{N_2(t)}} \qquad \forall t
\end{aligned} \tag{2.14}$$

Let's define:

$$f(P_1, C^*, t) = C^* - \log \frac{1 + P_1(e^{C^*} - 1)}{1 + \frac{P_1}{N_1(t)} + \frac{P_1}{N_2(t)}}$$

For example for the simple case of $m = 1$, the $\mathbf{U}$ matrix has the form:

$$\mathbf{U} = \begin{bmatrix} \sqrt{P_1} e^{j\theta_1} & \sqrt{1 - P_1} e^{j\theta_2} \\ \sqrt{P_1} e^{j\theta_3} & \sqrt{1 - P_1} e^{j\theta_4} \end{bmatrix}$$

and the condition in (2.14) reduces to:

$$\log \left[(1 + \frac{1}{N_1(t)})(1 + \frac{1}{N_2(t)}) - \frac{P_1^2 + (1 - P_1^2) + 2P_1(1 - P_1)\cos(\theta_1 - \theta_3 + \theta_4 - \theta_2)}{N_1(t)N_2(t)}\right] \geq f(P_1, C^*, t)$$

$$\forall t$$

$$\Rightarrow \log \left[e^{C^*} - \frac{P_1^2 + (1 - P_1^2) + 2P_1(1 - P_1)\cos(\theta_1 - \theta_3 + \theta_4 - \theta_2)}{N_1(t)N_2(t)}\right] \geq f(P_1, C^*, t) \qquad \forall t$$

The above constraint only depends on a linear combination of θ's namely $\theta_1 - \theta_3 + \theta_4 - \theta_2$ and hence once we find one set of $\theta's$, we can find infinitely many to satisfy the constraint and have exactly the same efficiency which is the best possible efficiency for any m. Note that these solutions generate different $\mathbf{U}$'s. From this equation, we can also easily see how to pick bad choices of θ's. For example, if $\theta_1 = \theta_3$ and $\theta_2 = \theta_4$, then we only achieve a total rate equal to that of MRC.

As we have more and more layers, the dither matrix has to satisfy more and more constraints for this upper bound to be tight which may not be possible. The main value of this result is that if we can get to this upper bound using a certain number of degrees of freedom, we can no longer improve the efficiency by increasing the degrees of freedom and using larger unitary matrices.

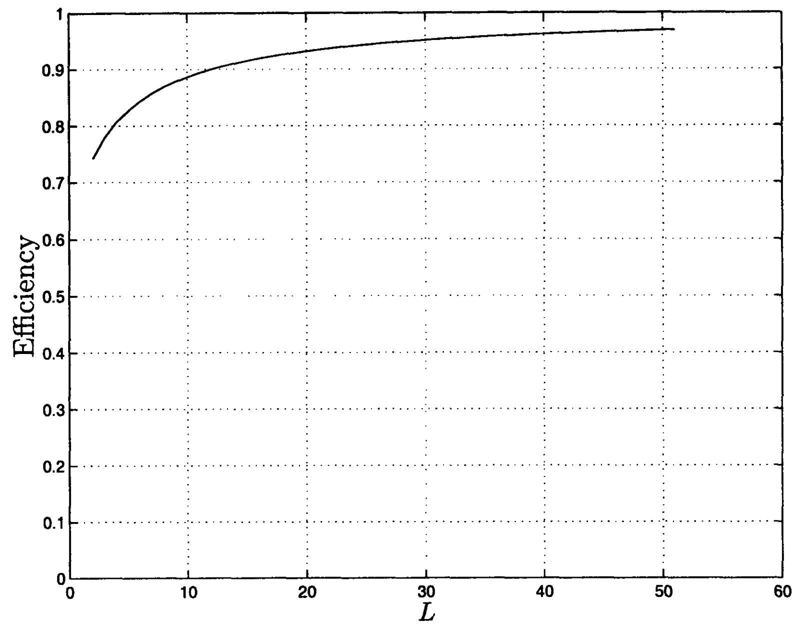


Figure 2-2. Upper-bound on efficiency for $C^* = 4.33$ bits (3 nats) as a function of the number of layers.

■ 2.4 Optimization Results

In this section, we present numerical solutions to the optimization design problem in (2.7) for two sample C^* values. To solve the optimization problem, we used a parameterization of the unitary matrices with m^2 real parameters derived in [2] and MATLAB's fminimax function. The parameters of the unitary matrices are all angles in the range of $[-\pi, \pi]$. In addition we have $2L - 2$ power allocation parameters over which to optimize. The goal is to examine the effect of the number of layers, L, and the number of symbols grouped together, m, on the overall efficiency. From the previous section, we can find an upper bound on the efficiency as a function of L which is independent of m. Hence if we can get to this upper bound with a certain dither dimension or m, then there will be no need for additional degrees of freedom, i.e., larger m.

■ 2.4.1 Reference for Comparison

To be able to make a fair judgment about the efficiency performance of the coding scheme and the effect of layering and grouping on this performance, we need to compare it to a reference scheme without any layering or grouping. As a reference, we will compare the efficiency results to the efficiency resulting only from repetition of the message over the

Table 2.1. Efficiency performance for $C^* = 4.33$ bits (3 nats) for different number of layers using only 1 dimensional complex dithers, $m = 1$. The reference efficiency for this C^* value and also the upper bound for different number of layers is shown for comparison. Also, the ratio of the minimum total rate achieved to capacity in each case and the corresponding power allocation is given. Using only 1 dimensional complex dithers can make the upper bound tight. Increasing the number of layers improves the efficiency up to a certain point after which the efficiency saturates and cannot get to the upper bound.

	Ref.	$L = 2$	$L = 3$	$L = 4$	$L = 5$	$L = 6$
Upper Bound	0.6916	0.7419	0.7788	0.8060	0.8269	0.8435
Opt. Eff.	0.6916	0.7419	0.7788	0.8060	0.8269	0.8278
$\min_t \sum_l R_l(t, U)/C^*$		0.9638	0.8998	0.8929	0.9346	0.9785
Power Alloc.		$\begin{bmatrix} 0.2934 \\ 0.7066 \end{bmatrix}$	$\begin{bmatrix} 0.1693 \\ 0.2613 \\ 0.5694 \end{bmatrix}$	$\begin{bmatrix} 0.1192 \\ 0.1425 \\ 0.2608 \\ 0.4774 \end{bmatrix}$	$\begin{bmatrix} 0.0922 \\ 0.0929 \\ 0.1526 \\ 0.2506 \\ 0.4116 \end{bmatrix}$	$\begin{bmatrix} 0.0736 \\ 0.0750 \\ 0.1030 \\ 0.1559 \\ 0.2358 \\ 0.3567 \end{bmatrix}$

two sub-channels with independent Bernoulli (1/2) dithering but without any layering or grouping of symbols. In the reference scheme, we just repeat the message over the two sub-channels with random independent dithering and then use MRC to combine the copies and decode the message. The achievable total rate in this reference scheme as a function of the channel realization parameter, t, is:

$$R_{\text{rep}} = \log(1 + \frac{1}{N_1(t)} + \frac{1}{N_2(t)})$$

This function has its minimum at $t = 0$ (see 2.11). Hence the efficiency of repetition alone is just:

$$\eta_{\text{rep}} = \frac{\log(1 + 2(e^{\frac{C^*}{2}} - 1)}{C^*} \tag{2.15}$$

2.4.2 Numerical Results

Here we present the numerical results for two sample C^* values: $C^* = 4.33$ bits (3 nats) and $C^* = 5.77$ bits (4 nats). These results are shown in Tables 2.1 and 2.2 respectively. The efficiency of the reference scheme is also shown in these tables for comparison. From these results we can make the following observations:

- Using only one complex degree of freedom ($m = 1$) achieves the upper bound up to $L = 5$.

Table 2.2. Efficiency performance for $C^* = 5.77$ bits (4 nats) for different number of layers using only 1 dimensional complex dithers, $m = 1$. The reference efficiency for this C^* value and also the upper bound for different number of layers is shown for comparison. Also, the ratio of the minimum total rate achieved to capacity in each case and the corresponding power allocation is given. Using only 1 dimensional complex dithers can make the upper bound tight. Increasing the number of layers improves the efficiency up to a certain point after which the efficiency saturates and cannot get to the upper bound.

	Ref.	$L = 2$	$L = 3$	$L = 4$	$L = 5$	$L = 6$
Upper Bound	0.6558	0.6942	0.7268	0.7527	0.7735	0.7908
Opt. Eff.	0.6558	0.6942	0.7268	0.7527	0.7735	0.7830
$\min_t \sum_l R_l(t,U)/C^*$		0.8024	0.8892	0.8958	0.8925	0.9924
Power Alloc.		$\begin{bmatrix} 0.2355 \\ 0.7645 \end{bmatrix}$	$\begin{bmatrix} 0.1280 \\ 0.2399 \\ 0.6321 \end{bmatrix}$	$\begin{bmatrix} 0.0879 \\ 0.1196 \\ 0.2538 \\ 0.5388 \end{bmatrix}$	$\begin{bmatrix} 0.0670 \\ 0.0734 \\ 0.1363 \\ 0.2532 \\ 0.4700 \end{bmatrix}$	$\begin{bmatrix} 0.0536 \\ 0.0539 \\ 0.0865 \\ 0.1458 \\ 0.2458 \\ 0.4143 \end{bmatrix}$

- For $L \geq 6$ the upper bound can not be achieved no matter how large m is. Using larger m's, i.e. $m \geq 2$, does not increase the efficiency for these L's.

- After using a certain number of layers, $L = 6$ for the above two examples, the efficiency saturates and cannot be made larger no matter how many more layers we have or how many extra dimensions we use in the dither matrices .

From the above observations we can see that grouping more symbols does not help in increasing the efficiency, at least not by any significant amount. In these examples, increasing m does not help at all since using $m = 1$ or only one dimensional complex dithers is enough to achieve the upper bound and we cannot do better than that.

Also, after using a few number of layers, the achievable efficiency saturates and cannot be made larger no matter how many extra layers or how many dimensions we use for our dither matrices. Even at this saturation region, increasing the dimension of the unitary matrices does not help and one dimensional complex dithers perform just as well.

Figures 2-3 to 2-5 show the layer rates and total rate as a function of t for $L = 2, 5$, and 8. Note that the realization with $t = \pm C^*/2$ is the realization in which one channel is off or in other words, $N_1(t)$ takes either its minimum value or is infinity and the realization with $t = 0$ corresponds to the case with $N_1(t) = N_2(t)$. We can clearly see that at $L = 8$, the condition to have a tight upper bound, that is to have the minimum rate of all layers except 1 at $t = \pm C^*/2$, is not satisfied since a few of the layers other than layer 1 also attain their

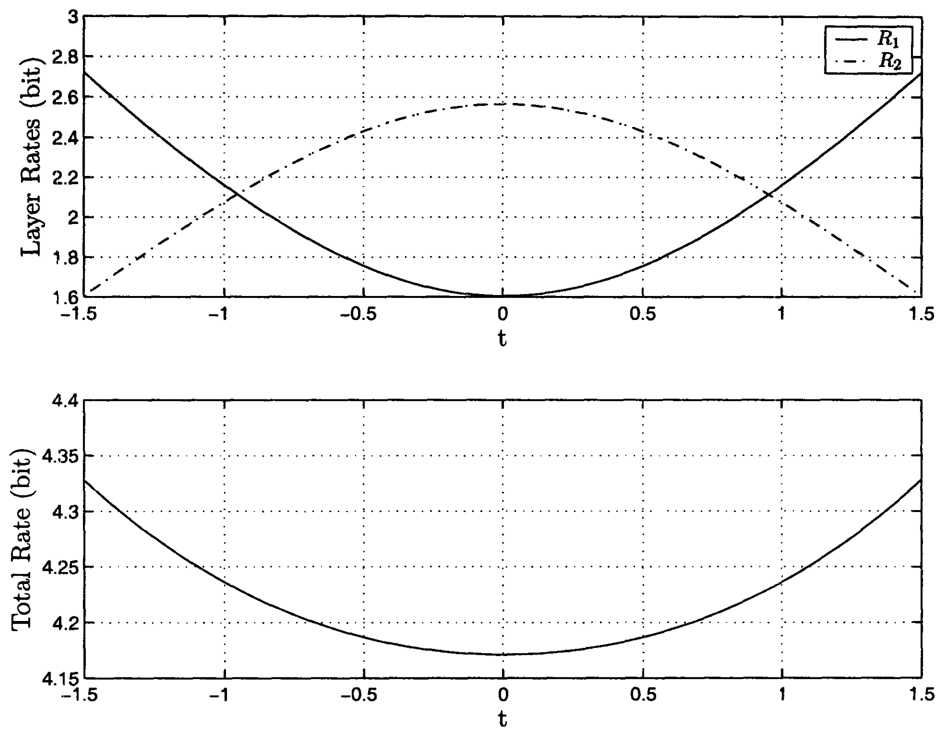


Figure 2-3. Layer rates and total rate vs. t for $C^* = 4.33$ bits (3 nats) and using $L = 2$ and $m = 1$. The efficiency is 74.19%.

minimum at $t = 0$.

To understand the saturation effect observed as we increase the number of layers, the limit on the efficiency when $L \to \infty$ should be examined. This will help us understand the ultimate upper bound on the efficiency for any C^* regardless of how many layers there are or how large the dither matrices are.

■ 2.5 Ultimate Upper Bound on Efficiency

Here we study the saturation effect on the efficiency that was observed after using a certain number of layers. The goal is to find the ultimate efficiency of the scheme as a function of the maximum rate.

■ 2.5.1 MMSE vs. MRC in the Limit of Large L

To see how the efficiency behaves as we increase the number of layers, we take the limit as $L \to \infty$. At a very large number of layers and hence a very low SNR per layer, the performance of the MMSE receiver is the same as the MRC receiver. To see this, note that

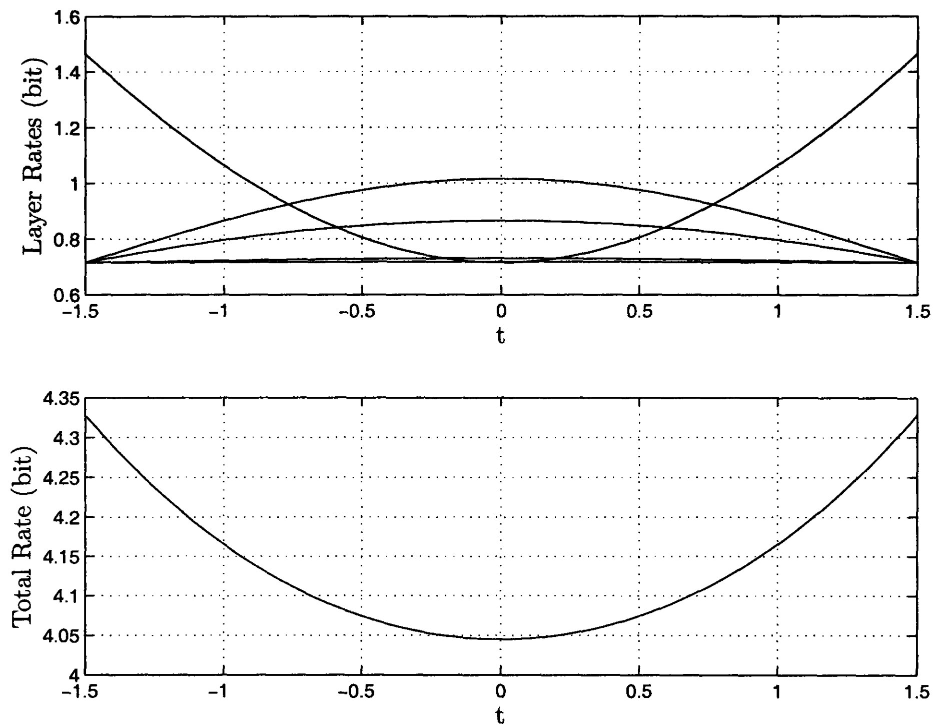


Figure 2-4. Layer rates and total rate vs. t for $C^* = 4.33$ bits (3 nats) and using $L = 5$ and $m = 1$. The efficiency is 82.69%.

the achievable rate for the lth layer using the MRC receiver is given by:

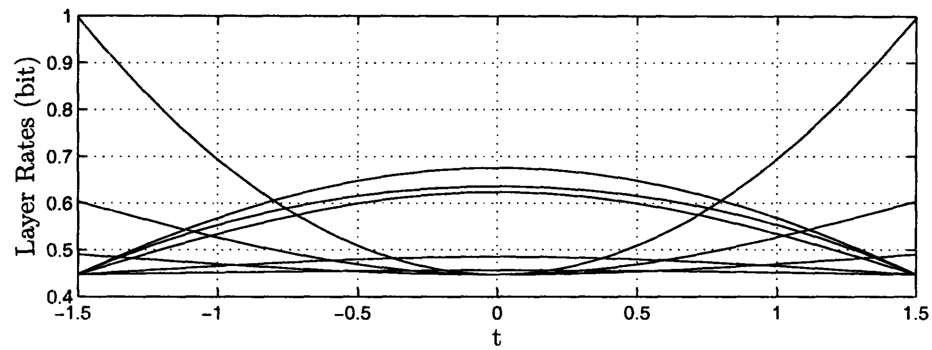
$$\log(1 + \mathrm{SNR}_1(l) + \mathrm{SNR}_2(l))$$

where $\mathrm{SNR}_1(l)$ is the SNR of layer l in the first sub-channel given by:

$$\mathrm{SNR}_1(l) = \frac{P_l}{\sum_{j=1}^{l-1} P_j + N_1}$$

and $\mathrm{SNR}_2(l)$ is defined symmetrically. In the limit of large L, SNR per layer is very small. Using the Taylor series expansion of the log function around 0, we know that for small x:

$$\log(1 + x) \approx x$$



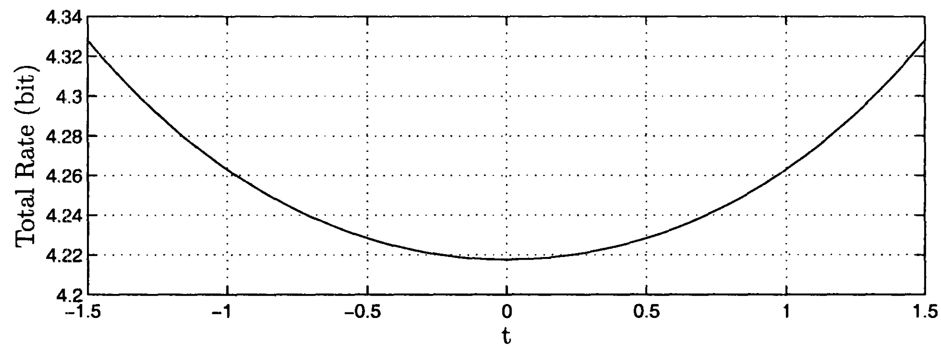


Figure 2-5. Layer rates and total rate vs. t for $C^* = 4.33$ bits (3 nats) and using $L = 8$ and $m = 1$. The efficiency is 82.78% that is the saturation efficiency. Note that there are a few layers other than 1 that attain their minimum at the channel realization with $t = 0$.

to a first order approximation. Therefore we can make the first order approximation:

$$\begin{aligned}\log(1 + \mathrm{SNR}_1(l) + \mathrm{SNR}_2(l)) &\approx \mathrm{SNR}_1(l) + \mathrm{SNR}_2(l) \\ &\approx \log(1 + \mathrm{SNR}_1(l)) + \log(1 + \mathrm{SNR}_2(l))\end{aligned}$$

which is the best possible achievable mutual information and hence at low SNR is the same as the achievable rate using the MMSE receiver. Hence, to find the efficiency in the limit of large L, we can assume an MRC receiver in our analysis.

■ 2.5.2 Behavior of the Layer Rate Function in the Limit of Large L

To find the asymptotic maximum achievable rate, we should examine the behavior of the rate function for different layers and in the limit of large L. For any layer, the minimum rate happens either at a channel realization with $N_1 = N_2 = 1/(e^{C^*/2} - 1)$ or $N_2 = \infty$ and $N_1 = 1/(e^{C^*} - 1)$ (or vice versa but because of complete symmetry we only consider this case). This is again because the function:

$$\log(1 + \frac{P_l}{\sum_{j=1}^{l-1} P_j + N_1(t)} + \frac{P_l}{\sum_{j=1}^{l-1} P_j + N_2(t)})$$

has a zero first derivative at $t = 0$ which changes sign at this point. Let's define:

$$N_{\text{eq}} = \frac{1}{e^{\frac{C^*}{2}} - 1} \tag{2.16}$$

$$N_{\infty} = \frac{1}{e^{C^*} - 1} \tag{2.17}$$

Note that $N_{\infty} < N_{\text{eq}}$. The realization at which a layer attains its minimum depends on the amount of interference seen by that layer. For an arbitrary layer, the achievable rate when $N_1 = N_2$ is given by:

$$R_{\text{eq}}(l) = \frac{P_l}{\sum_{j=1}^{l-1} P_j + N_1} + \frac{P_l}{\sum_{j=1}^{l-1} P_j + N_2} = \frac{2P_l}{\sum_{j=1}^{l-1} P_j + N_{\text{eq}}}$$

and the achievable rate when $N_2 = \infty$ and $N_1 = N_{\infty}$ is given by:

$$R_{\infty}(l) = \frac{P_l}{\sum_{j=1}^{l-1} P_j + N_1} = \frac{P_l}{\sum_{j=1}^{l-1} P_j + N_{\infty}}$$

We can clearly see that for a certain layer, whether R_{∞} is the minimum achievable rate or R_{eq}, depends on the interference power in all the other layers, i.e., $\sum_{j=1}^{l-1} P_j$. Define P_c as the accumulated interference power which is the turning point. To find P_c we solve for:

$$\begin{aligned}
& R_{\text{eq}}(l) = R_{\infty}(l) \\
\Rightarrow \quad & \frac{2P_l}{\sum_{j=1}^{l-1} P_j + Neq} = \frac{P_l}{\sum_{j=1}^{l-1} P_j + N_{\infty}} \\
\Rightarrow \quad & \frac{2}{P_c + Neq} = \frac{1}{P_c + N_{\infty}} \\
\Rightarrow \quad & P_c = N_{\text{eq}} - 2N_{\infty} \\
\Rightarrow \quad & P_c = \frac{1}{e^{\frac{C^*}{2}} - 1} - \frac{2}{e^{C^*} - 1}
\end{aligned}$$

Now if for $\epsilon > 0$, $\sum_{j=1}^{l-1} P_j = P_c - \epsilon$, then:

$$R_\infty(l) = \frac{P_l}{N_{\text{eq}} - N_\infty - \epsilon}$$
$$R_{\text{eq}}(l) = \frac{P_l}{N_{\text{eq}} - N_\infty - \frac{\epsilon}{2}}$$

and $R_\infty(l) > R_{\text{eq}}(l)$. Hence for a layer l where $\sum_{j=1}^{l-1} P_j < P_c$, the minimum rate happens when $N_1 = N_2 = N_{\text{eq}}$ and this rate is the limiting rate for the coding scheme and for a layer l where $\sum_{j=1}^{l-1} P_j > P_c$, the minimum rate happens at $N_1 = N_\infty$ and $N_2 = \infty$ and this is the limiting rate.

2.5.3 Total Achievable Rate in the Limit of Large L

To find the total achievable rate we should sum over the minimum achievable rates of all layers. Let's define L^* as the layer for which $\sum_{l=1}^{L^*-1} P_l = P_c$. In the limit of large L or small SNR per layer, using the results of the previous section, the total achievable rate is given by:

$$\begin{aligned}
R_{\text{tot}} &= \sum_{l=1}^{L^*-1} \log(1 + \frac{2P_l}{\sum_{j=1}^{l-1} P_j + N_{\text{eq}}}) + \sum_{l=L^*}^{L} \log(1 + \frac{P_l}{\sum_{j=1}^{l-1} P_j + N_\infty}) \\
&\approx \sum_{l=1}^{L^*-1} 2\log(1 + \frac{P_l}{\sum_{j=1}^{l-1} P_j + N_{\text{eq}}}) + \sum_{l=L^*}^{L} \log(1 + \frac{P_l}{\sum_{j=1}^{l-1} P_j + N_\infty}) \\
&= 2\log(1 + \frac{P_c}{N_{\text{eq}}}) + \log(1 + \frac{1 - P_c}{P_c + N_\infty}) \\
&= 2\log(1 + \frac{N_{\text{eq}} - 2N_\infty}{N_{\text{eq}}}) + \log(1 + \frac{1 - N_{\text{eq}} + 2N_\infty}{N_{\text{eq}} - N_\infty})
\end{aligned} \tag{2.18}$$

where N_{eq} and N_∞ are functions of C^* only and are given by (2.16) and (2.17). Hence, using the maximum achievable rate from (2.18), the upper bound on efficiency as $L \to \infty$ is:

$$\eta_{\max} = \frac{R_{\text{tot}}(C^*)}{C^*} \tag{2.19}$$

Note that in our derivation, we find the maximum achievable rate assuming that we can use a different rate for each layer which is equal to its worst case channel realization achievable rate. This is off course better than using a common rate for all layers equal to their minimum.

Figure 2-6 shows the upper bound on achievable efficiency as a function of C^*. It also shows the efficiency of the reference scheme in (2.15), i.e., the efficiency if we just repeat the message over the two sub-channels and then use MRC to combine them.

Taking the limit of the upper bound in (2.19) as $C^* \to \infty$ and $C^* \to 0$ we get:

$$\begin{aligned} \lim_{C^* \to \infty} \eta_{\max} &= \frac{1}{2} \\ \lim_{C^* \to 0} \eta_{\max} &= 1 \end{aligned}$$

Also for the reference efficiency in (2.15) we have:

$$\begin{aligned} \lim_{C^* \to \infty} \eta_{\text{rep}} &= \frac{1}{2} \\ \lim_{C^* \to 0} \eta_{\text{rep}} &= 1 \end{aligned}$$

Hence, we conclude that the coding scheme performs close to simple repetition at the limit of large C^*'s. The maximum gain we get from simple repetition is at $C^* = 3.76$ bits (2.61 nats) and is equal to 13.75%.

As expected from Tables 2.1 and 2.2, the ultimate upper bound for $C^* = 4.33$ bits is 82.78% and for $C^* = 5.77$ bits is 78.31%, equal to the saturation efficiencies observed. We can get to these upper bounds using the MMSE receiver with $L = 6$ layers only.

2.6 Effect of Dither Dimension on the Efficiency: Real vs. Complex vs. Complex Unitary

Here we examine the effect of the dither dimension on efficiency. As we observed in the numerical results (Tables 2.1 and 2.2), only a single complex degree of freedom (a one dimensional complex dither) for C^*'s tested is enough to make the upper bound tight. This means that going beyond a single complex dimension will not improve the efficiency at least not by any noticeable amount.

It is interesting to investigate the efficiency performance using only real dithers, i.e., only ± 1's. Using ± 1 dithers, there is only a finite number of possibilities (2^{2L}) for the overall dither matrix not considering the power allocation degrees of freedom. We searched over these choices of dither matrices with the optimal power allocation found in the case

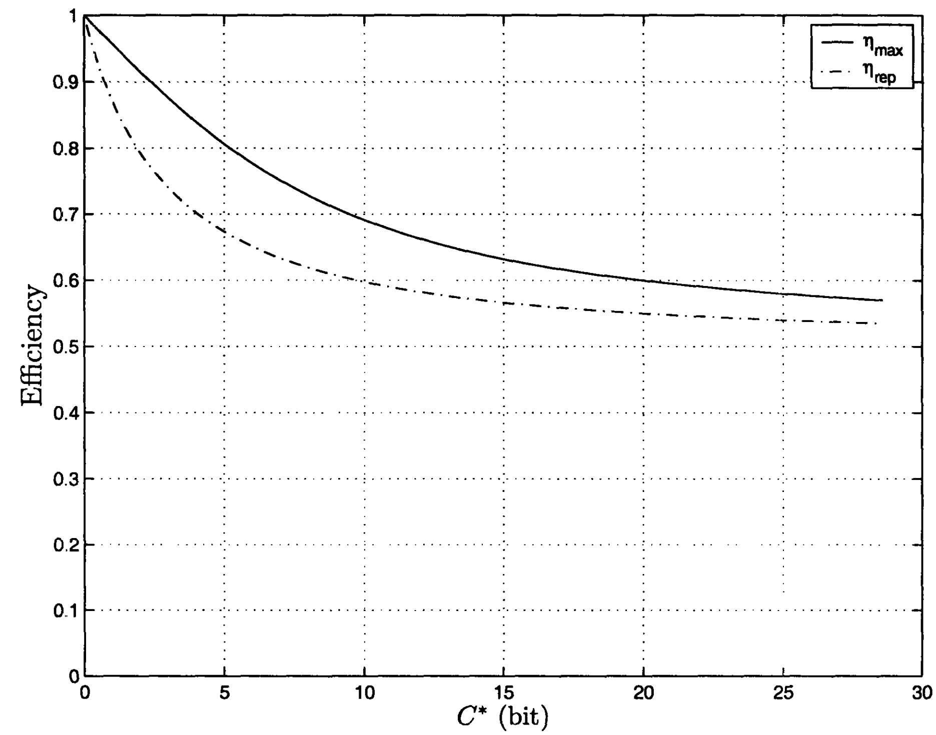


Figure 2-6. Ultimate upper bound on efficiency of the coding scheme as a function of C^*. The efficiency of the reference scheme is also shown for comparison.

of the complex dithers for the two C^*'s tested (Tables 2.1 and 2.2) and compared the best possible efficiency with the optimal efficiency obtained using the complex dithers. Tables 2.3 and 2.4 show the results in comparison with the case of complex dithers.

From these results we observe that real dithers perform just as well as complex dithers. This is what we expect since it was already concluded that going to higher complex dimensions does not help with the efficiency and this implies that adding another degree of freedom to a real dither to make it complex does not make any noticeable change to efficiency either.

In short adding degrees of freedom to the dither matrices does not help in increasing the efficiency performance and real dithers, ± 1's, work just as well.

■ 2.7 Effect of 1 Bit of Channel Side Information at the Transmitter (CSIT)

As we saw in the pervious section, the efficiency gain of the coding scheme is not that high. The largest gain obtained is about 13.75% above the dithered scheme without any layering and with repetition over the sub-channels. It is therefore valuable to investigate different

Table 2.3. Efficiency performance of the coding scheme for $C^* = 4.33$ bits using real and complex 1 dimensional dithers. The upper bound on efficiency as a function of L is also shown. Real dithers perform just as well as complex dithers.

	Ref.	$L=2$	$L=3$	$L=4$	$L=5$
Upper Bound	0.6916	0.7419	0.7788	0.8060	0.8269
Opt. Eff., Complex Dither	0.6916	0.7419	0.7788	0.8060	0.8269
Eff., Real Dither	0.6916	0.7419	0.7788	0.8060	0.8269

Table 2.4. Efficiency performance of the coding scheme for $C^* = 5.77$ bits using real and complex 1 dimensional dithers. The upper bound on efficiency as a function of L is also shown. Real dithers perform just as well as complex dithers.

	Ref.	$L=2$	$L=3$	$L=4$	$L=5$
Upper Bound	0.6558	0.6942	0.7268	0.7527	0.7735
Opt. Eff., Complex Dither	0.6558	0.6942	0.7268	0.7527	0.7735
Eff., Real Dither	0.6558	0.6942	0.7268	0.7527	0.7735

additional improvements to the scheme to increase this gain. In this section we investigate the improvement obtained as a result of 1 bit of CSIT. The 1 bit information here tells us which sub-channel has a higher quality.

2.7.1 Code Design Formulation with 1 Bit of CSIT

The code design optimization problem in the presence of 1 bit of CSIT is very similar to the general case. Without loss of generality we can assume that $N_1 \leq N_2$, i.e., sub-channel 1 has a higher SNR. Here we do not consider non-uniform power allocation across the sub-channels. We still assume that the same power is used over both sub-channels to study the gain obtained only from using an asymmetric power allocation across the layers of the two sub-channels and separate the two effects. Having this new piece of information, we should now optimize the dither matrix over a different range. The optimization problem in (2.7) is still valid only the range of t over which we optimize is now changed to $-C^*/2 \leq t \leq 0$ (instead of $|t| < C^*/2$).

2.7.2 Numerical Results

The modified code design optimization problem was solved for the two C^* values as before, namely $C^* = 4.33$ bits and $C^* = 5.77$ bits. Table 2.5 shows the efficiency resulting from

Table 2.5. Efficiency performance of the coding scheme with 2 layers and using 1 bit of CSIT and its comparison to the case with no CSIT. The reference efficiency and the upper bound from (2.19) are also shown for comparison. Using a single bit of CSIT we obtain a significant improvement even using only 2 layers. Note that the power allocation is no longer symmetric.

$L = 2$, $m = 1$	4.33 bits	5.77 bits
Eff., 1 bit CSIT	0.9059	0.8732
Optim. Eff., No CSIT	0.7419	0.6942
Reference	0.6916	0.6558
Ultimate Upper Bound	0.8278	0.7831
Power Alloc. Sub-chan 1	$\begin{bmatrix} 0.2180 \\ 0.7820 \end{bmatrix}$	$\begin{bmatrix} 0.1590 \\ 0.8410 \end{bmatrix}$
Power Alloc. Sub-chan 2	$\begin{bmatrix} 0.9330 \\ 0.0670 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

the optimized dither matrix for the problem with 1 bit CSIT.

Looking at the optimization results, we can see that there is a significant gain in the efficiency only using 2 layers. This gain is not only over the efficiency of the scheme with no CSIT with 2 layers but also on the ultimate upper bound on efficiency in (2.19) which is independent of L. The gain over the upper bound using only 2 layers at $C^* = 5.77$ bits is 9% and at $C^* = 4.33$ bits is 8%. The gain over the scheme with no CSIT with the same 2 layers at $C^* = 5.77$ bits is 17.9% and at $C^* = 4.33$ bits is 16.4%. These are significant improvements especially since we are only using a single bit of information.

The reason for these improvements is that now the range over which the dither matrix is optimized is half the size. Also, this bit of information allows us to use an asymmetric power allocation. The new power allocation strategy acts as this: In the better sub-channel, a small amount of power is allocated to the first layer with no interference and the rest to the second layer. In the bad sub-channel, most of power is allocated to the first layer that sees no interference since the sub-channel is low SNR already.

Figures 2-7 and 2-8 show the optimized layer rates and total rate in the range of t. (Here we assumed $N_1 \leq N_2$, i.e., $t \leq 0$, without loss of generality). Again increasing the dimension of the dither matrices beyond 1 does not improve the efficiency. However, having more layers will result in higher gains.

There are two situations in which we can use the new optimized dither matrix with this type of channel information. If we are designing for several different users in which case any realization of the parallel channel is possible, we can use this new dither matrix along

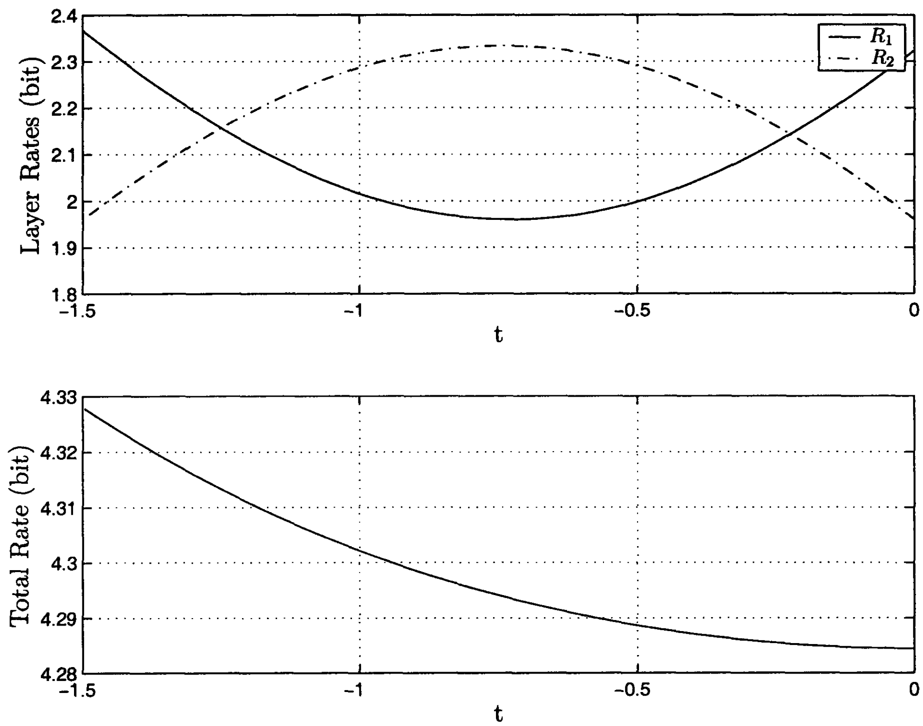


Figure 2-7. Layer rates and total rate vs. t for $C^* = 4.33$ bits (3 nats) using a single bit of CSIT and with $L = 2$ and $m = 1$. The efficiency is 90.59%.

with 1 bit CSIT per user. If for a user, $N_1 \leq N_2$, then we use the above optimized dither matrix and if $N_1 \geq N_2$ then we just flip the inputs into the sub-channels.

The other situation arises when we are designing for a single user whose channel realization, even though unknown and time-varying, is such that most of the time one known sub-channel is at a higher SNR than the other one. In that case, we can just use this dither matrix which is optimized to the range of interest having just an initial bit of CSIT.

■ 2.8 Optimizing the Code Design to Different Ranges of Channel Realization

The 1 bit CSIT in the previous section conveys information about the asymmetry of the channel. In other words, it tells us which sub-channel is of higher SNR. Hence the optimization is done on an asymmetric subset of the overall range in which one of the sub-channels is stronger. In this section we examine the effect of different types of partial channel information on the performance of the code. Specifically, we will look at the case where we know that the sub-channel qualities are similar.

We may be dealing with situations in which the sub-channel conditions are close to each

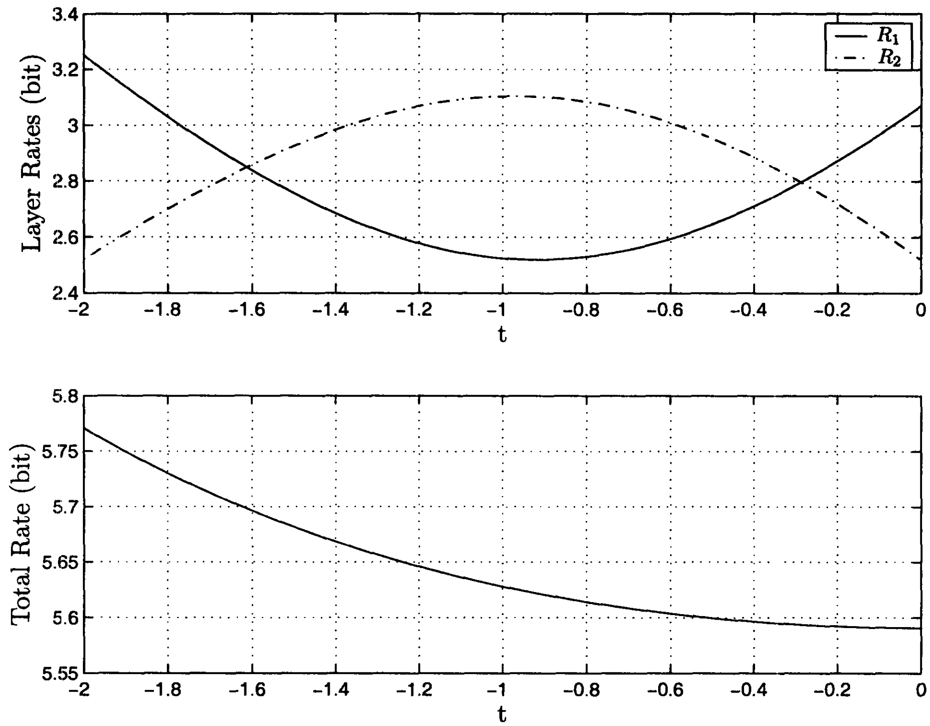


Figure 2-8. Layer rates and total rate vs. t for $C^* = 5.77$ bits (4 nats) using a single bit of CSIT and with $L = 2$ and $m = 1$. The efficiency is 87.32%.

other. This means that even though any of the two sub-channels can be stronger than the other one, they will not be very far apart.

We can again look at this situation in two ways: A single time-varying user whose channel realization has the above property all the time, or several users whose channel realizations have the above property. For the time-varying single user channel case, we need constant or frequent CSIT to use the dither matrix of previous section since at any time the ordering of the sub-channels might change. For the multiple user scenario, we need 1 bit of CSIT per user for a similar reason. Hence, in these situations, it is not optimal to optimize over an asymmetric range. In such cases, we should change the optimization range to a subset which is symmetric with respect to both sub-channels. This way, very little information will be required at the transmitter. Note that this scheme works only if we are designing for users whose sub-channel qualities do not get too far apart.

Let us define:

$$d = N_{\text{eq}} - N_{\infty}$$

where N_{eq} and N_{∞} are defined in (2.16) and (2.17). For some $k > 1$, we formulate the

Table 2.6. Efficiency performance of the coding scheme for $C^* = 4.33$ bits, where the dither matrix is optimized to a middle range of realizations. Here, $N_\infty + d/k \le N_1(t) \le \gamma(k)$, where $k = 4$ and $\gamma(k)$ is the corresponding value for $N_1(t)$ when $N_2(t) = N_\infty + d/k$.

	Ref.	$L = 2$	$L = 3$
Opt. Eff., Middle Range	0.6916	0.9016	0.9221
Opt. Eff., Entire Range	0.6916	0.7419	0.7788

Table 2.7. Efficiency performance of the coding scheme for $C^* = 5.77$ bits, where the dither matrix is optimized to a middle range of realizations. Here, $N_\infty + d/k \le N_1(t) \le \gamma(k)$, where $k = 4$ and $\gamma(k)$ is the corresponding value for $N_1(t)$ when $N_2(t) = N_\infty + d/k$.

	Ref.	$L = 2$	$L = 3$
Opt. Eff., Middle Range	0.6558	0.8893	0.9107
Opt. Eff., Entire Range	0.6558	0.6942	0.7268

new problem by optimizing over the range of $N_\infty + d/k \le N_1(t) \le \gamma(k)$ where $\gamma(k)$ is the corresponding value for $N_1(t)$ when $N_2(t) = N_\infty + d/k$. Equivalently, we optimize over a symmetric range of $|t| < \alpha(k)$ for $\alpha(k) = |C^*/2 - \log(1 + 1/(N_\infty + d/k)|$. We can easily move between these equivalent representations using the parameterization of the noises in (2.2) and (2.3). Tables 2.6 and 2.7 show the efficiency results of this optimization for $k = 4$, i.e., when the two sub-channels are assumed not to take realizations in an interval of length $d/4$ at the extreme ends of the range for N_1 and N_2. Figure 2-9 shows the layer rates and total rate for the case of $C^* = 4.33$ bits and $L = 3$.

We again observe a significant gain over the case where we optimize over the entire range, i.e., where we assume no knowledge of channel at all. This gain is over both the upper bound on the efficiency of the scheme with no channel knowledge in (2.19) and its efficiency using the same number of layers. At $C^* = 4.33$ bits and using 2 layers we have a gain of 16% with this limited channel knowledge compared to the scheme with no channel knowledge and 2 layers and a gain of 7.4% over its upper bound. At $C^* = 5.77$ bits and using 2 layers we have a gain of 19.5% compared to the scheme with no channel knowledge and 2 layers and a gain of 10.6% over its upper bound. Note that this is a significant gain especially since we are only using 2 layers and very limited knowledge of the channel.

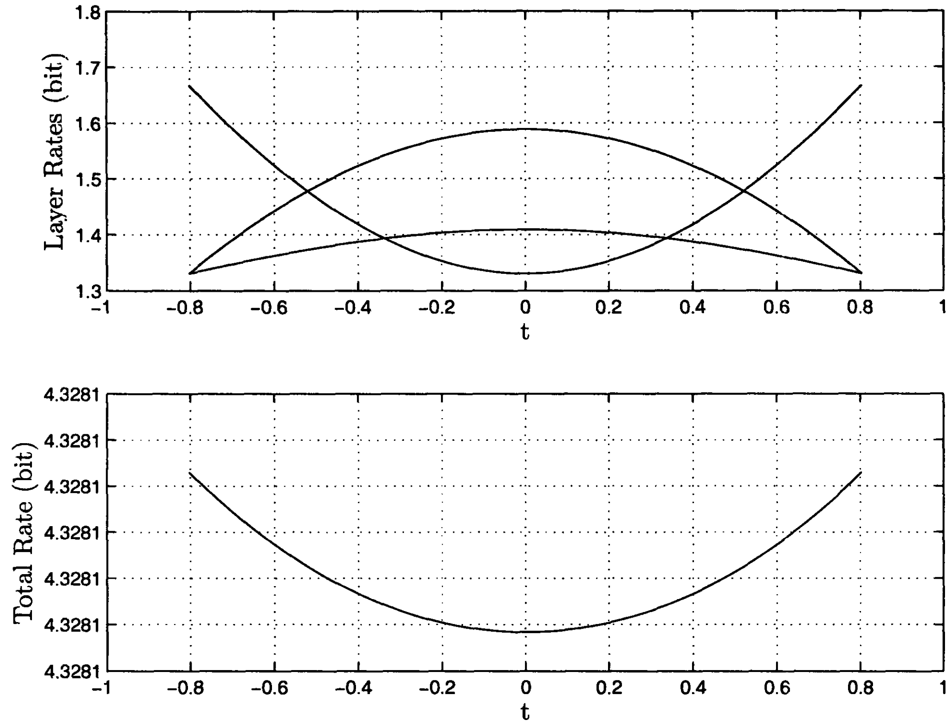


Figure 2-9. Layer rates and total rate vs. t for $C^* = 4.33$ bits (3 nats) using $L = 3$ and $m = 1$, where the dither matrix is optimized to a middle range of realizations. Here, $N_\infty + d/k \leq N_1(t) \leq \gamma(k)$, where $k = 4$ and $\gamma(k)$ is the corresponding value for $N_1(t)$ when $N_2(t) = N_\infty + d/k$. The resulting efficiency is 92.21%.

2.9 Robustness to Channel Knowledge

In the previous sections, we examined the performance of the coding scheme in the presence of limited channel knowledge. We can also look at these situations in terms of robustness of the scheme to unreliable channel knowledge. Assume that partial CSIT on sub-channel gains is available but that this information is not reliable. In the range around $N_1 = N_2$, using the results listed in Tables 2.6 and 2.7, we can see that the scheme is very robust to this unreliable channel information. As long as the channel information on sub-channel noises is correct within a range of $N_{\text{eq}} \pm 3d/4$ and using only 2 layers, we can get to an efficiency of 90% for $C^* = 4.33$ bits.

We can also look at the results of the scheme with 1 bit of CSIT in terms of robustness to channel knowledge in the range around $N_1 = N_\infty$ and $N_2 = \infty$ or vice versa. Again we are very robust to unreliable channel knowledge. In this range, as long as the channel knowledge is reliable enough to determine the stronger sub-channel, we can get an efficiency of 90% using only 2 layers in the case of $C^* = 4.33$ bits.

In short even a little bit of CSIT helps greatly in the parallel channel problem and the scheme is very robust to unreliable channel knowledge.

2.10 Decoder Structure and Its Effect on Performance

So far we studied the effect of layering, dither dimension and partial CSIT on the coding scheme. In all the analysis so far, we assumed to use an MMSE receiver, which is information lossless, combined with successive cancellation. It will therefore be interesting to examine the effect of decoder structure on the performance of the scheme. Here we study the effect of using a simple MRC receiver instead of the MMSE receiver. We will optimize the power allocations for the new MRC receiver for the case of coding with no CSIT and compare its performance to the optimized efficiency of the MMSE.

Before we present the new results, note that the MRC receiver does not make use of the full dither information. Decoding each layer, it just treats the lower level interfering layers as noise. We dither different layers in the two sub-channels independently using Bernoulli (1/2) dithers, therefore the interference in the repeated copies across the two sub-channels are uncorrelated and add up non-coherently. Note that here only random real dithers, i.e. ± 1's, are sufficient to make the interference uncorrelated. Hence the only optimization needed is that over the power allocation parameters.

2.10.1 Code Design Formulation

Let $P_l(1)$ and $P_l(2)$ denote the powers in the lth layer of the two sub-channels. Because of the symmetry of the problem, at the optimal solution, $P_l(1) = P_l(2) = P_l$. Using MRC, the achievable rate in the lth layer, $R_l(t)$, using a worst case Gaussian assumption of interference and the fact that the interference layers across the two subchannels are uncorrelated, is given by:

$$R_l(t) = \log(1 + \frac{P_l}{\sum_{j=1}^{l-1} P_j + N_1(t)} + \frac{P_l}{\sum_{j=1}^{l-1} P_j + N_2(t)})$$

Table 2.8. Efficiency comparison of MMSE vs. MRC for $C^* = 4.33$ bits.

	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$	Up. Bound
MMSE	0.7419	0.7788	0.8060	0.8269	0.8278	0.8278
MRC	0.7419	0.7735	0.7805	0.7911	0.7961	0.8278

Table 2.9. Efficiency comparison of MMSE vs. MRC for $C^* = 5.77$ bits.

	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$	Up. Bound
MMSE	0.6942	0.7268	0.7527	0.7735	0.7831	0.7831
MRC	0.6942	0.7268	0.7345	0.7426	0.7517	0.7831

Hence the optimization problem becomes:

$$\max_{P_1,\cdots,P_L} \min_t \min_l R_l(t)$$

$$\text{s.t.}$$

$$\sum_{l=1}^{L} P_l = 1$$

$$P_l \geq 0 \qquad \forall l$$

2.10.2 Optimization Results

Tables 2.8 and 2.9 show the optimization results for $C^* = 4.33$ bits and $C^* = 5.77$ bits respectively.

We can see that the performance of MRC is very close to that of MMSE. At a small number of layers, it performs basically the same as MMSE. By increasing the number of layers, it loses a bit of performance to MMSE. Also the differential gain from adding an extra layer for MRC becomes smaller as the total number of layers is increased.

Another observation is that the upper bound on efficiency with MMSE is reached at 6 layers whereas this is not the case for MRC. We need many more layers for MRC to be able to get to this upper bound. However, note that the gap between the performance of MRC at 5 or 6 layers is very close to that of MMSE and the upper bound and is only about $3-4\%$ below these.

■ 2.10.3 Reasoning and Conclusions

To see the reason for this close performance of MMSE and MRC in the case with no CSIT, let's look at the coding scheme with 2 layers for $C^* = 4.33$ bits. Figures 2-10 and 2-11 show the MMSE layer rates and total rate and the MRC layer rates and total rate, respectively. Note that since the power allocation is symmetric, all the layers in the MRC scheme take a maximum or a minimum at $t = 0$ or equivalently when $N_1(t) = N_2(t) = N_{\text{eq}}$. This can be seen by solving $\frac{\partial}{\partial t} R_l(t) = 0$. For the first layer as derived earlier in (2.11), this is a minimum. Also the rate in the first layer is the same, regardless of using MMSE or MRC since this layer sees no interference.

Now looking at the two figures, we can clearly see why the two decoder structures have the same performance. The achievable rate for layer 2 using the MRC receiver is lower than its achievable rate using the MMSE receiver at all channel realizations. However, the limiting rate here is the minimum rate of layer 1 which happens at $t = 0$. This implies that the MRC receiver will have the same performance for the parallel channel as MMSE, as long as the minimum achievable rate in layer 2 can be made as large as $R_1(t = 0)$ when using the same power allocation as that of the optimum MMSE. In other words, as long as the optimum power allocation in the MMSE case can equalize the minimum achievable rate of both layers in the MRC case, it is a valid power allocation for MRC and generates the same efficiency.

Note that it is true that the MMSE achieves higher total rates for any single channel realization or t, but the limiting factor in rate allocation happens at $t = 0$ and hence for the parallel channel without CSIT, the MRC receiver does just as well when it satisfies the above conditions.

As L increases, it becomes harder for MRC to be able to keep the minimum rate of all the layers the same as optimum MMSE, i.e., the optimum power allocation in MMSE does not work for MRC anymore. However, since the limiting factor in rate allocation is the minimum of layer 1 at $t = 0$ and not the overall performance in terms of achievable rate at all other channel realizations, the performance gap between the two decoder structures will be small.

Note that the results obtained here are based on a worst case Gaussian assumption of the interference which is reasonable at a large number of layers according to the central limit theorem since the layers are i.i.d.. However, at a small number of layers, this assumption

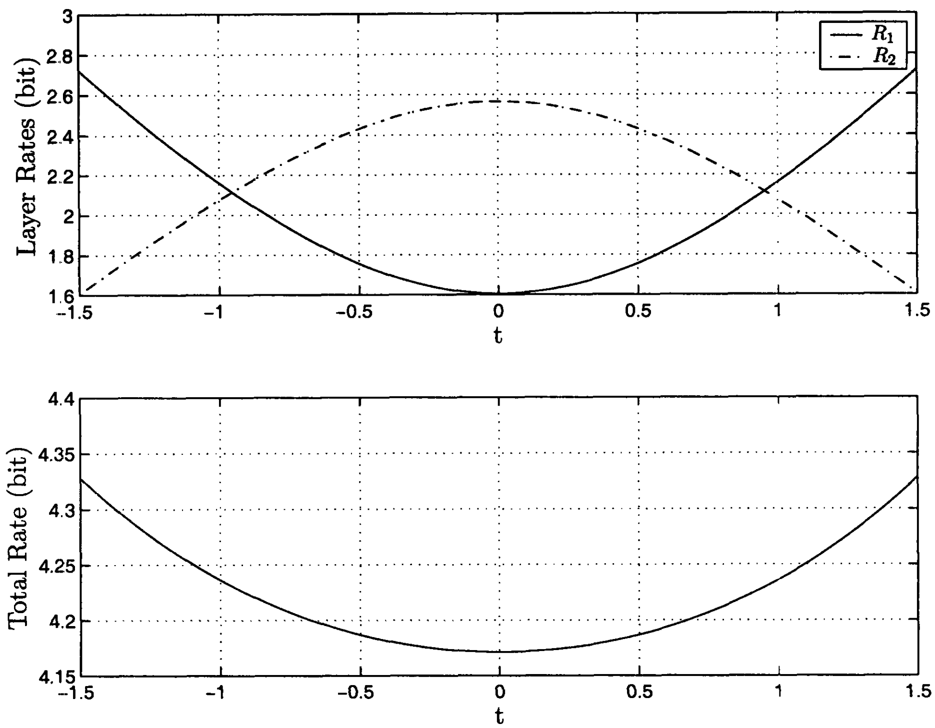


Figure 2-10. Layer rates and total rate vs. t using an MMSE receiver for $C^* = 4.33$ bits. Here $L = 2$ and $m = 1$. The MMSE efficiency is 74.19%.

is not reasonable. For example in the scheme with only 2 layers, the exact achievable rate of layer 2 at any t will be higher than that found by our approximation. But since the efficiency of MMSE gives us an upper bound on the achievable efficiency of MRC, and since the worst case performance of the MRC is already close to that of MMSE, we conclude that the exact performance of MRC is going to be very close to that of MMSE. An exact noise analysis combined with optimization is more complicated and will not be pursued here.

2.11 Summary

We can summarize the highlights of this chapter on the properties and performance of the layered coding scheme with deterministic dither as follows:

- Layering results in efficiency gains. However, the efficiency saturates after a certain number of layers and cannot be made larger.
- Increasing the number of degrees of freedom by using higher dimensional dither matrices does not have any noticeable effect on the efficiency performance. Basically a

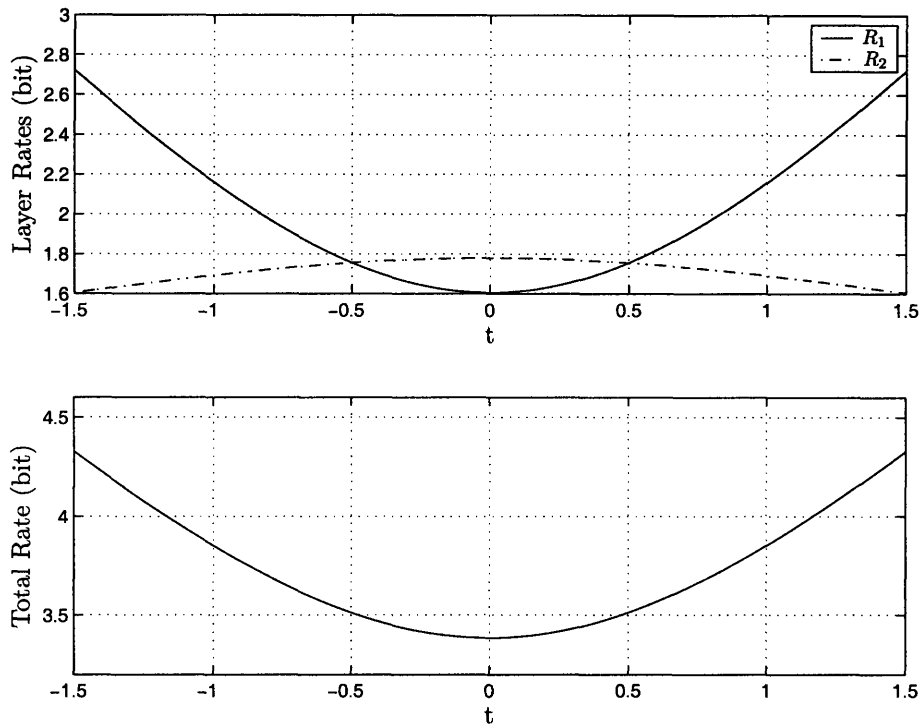


Figure 2-11. Layer rates and total rate vs. t using an MRC receiver for $C^* = 4.33$ bits. A Gaussian assumption is used for the interference. Here $L = 2$ and $m = 1$. The MRC efficiency is 74.19% which is the same as that for MMSE.

real ± 1 dither performs just as well.

- Partial CSIT (even a single bit) makes a significant improvement in the efficiency performance of the scheme.

- The coding scheme that exploits partial CSIT is very robust to unreliable channel knowledge.

- Using the MRC receiver in the coding scheme with no CSIT performs close to the MMSE receiver but will take many more layers to achieve the ultimate upper bound on efficiency for any C^*.

Chapter 3

A Sub-block Structure for Rateless Coding

In this chapter we will explore an alternative approach to universal code design for parallel Gaussian channels. We will simultaneously consider the rateless design of these codes since it does not complicate the treatment. If the rateless property is not required, these codes can still be used by removing the temporal redundancy as will become clear.

As discussed in Chapter 1, low complexity rateless codes using layering with non-uniform time-varying power allocation, random Bernoulli (1/2) dithering, and repetition, are designed in [3] that achieve high efficiencies if the base code rate is sufficiently low. The decoder structure uses an MRC receiver along with successive cancellation. The power allocation is designed such that the rates can be equalized across the layers and hence rate allocation is made easy and the same codebook is used for all layers.

The difference in the parallel Gaussian channel is that the rate allocation problem is not as easy since even for a given maximum rate, C^*, the noise pair in the channel is uncertain. Using only layering and repeating over the two sub-channels, the power allocation that enables a rate of C^*/L in each layer is given by:

$$r_l = \frac{2g(l) + \frac{1}{\mathrm{SNR}_1} + \frac{1}{\mathrm{SNR}_2}}{2} - \frac{\sqrt{[2g(l) + \frac{1}{\mathrm{SNR}_1} + \frac{1}{\mathrm{SNR}_2}]^2 - 4(g(l) + \frac{1}{\mathrm{SNR}_1})(g(l) + \frac{1}{\mathrm{SNR}_2})(1 - \frac{1}{e^{2C^*/L}})}}{2}$$

where P_l is the power in layer l, P is the total power per sub-channel, $\mathrm{SNR}_k = P/N_k, k \in \{1,2\}$, $r_l = P_l/P$, and $g(l) = [1-\sum_{k>l} r_k]$. Note that the problem is symmetric with respect to the two sub-channels and hence the power allocations must be symmetric as well. We can see that this power allocation depends on the specific SNR pair which is not known.

Thus, the challenge in the parallel Gaussian channel is to have a layered code in which we can assign rates to each layer that are achievable regardless of the specific SNR pair realized. One solution is to design a coding scheme that makes all the layers symmetric so that an equal rate can be assigned to all of them for any SNR pair and any number of repetitions, r (if rateless). This way the same rate and power can be used across layers and the rate and power allocation problem is solved.

This chapter will introduce and analyze such a scheme whose goal is to construct a code that achieves an effective rate close to capacity for any channel realization of the form:

$$\frac{1}{2}\log(1+\mathrm{SNR}_1(r))+\frac{1}{2}\log(1+\mathrm{SNR}_2(r))=\frac{C^*}{r} \qquad \forall r$$

and is symmetric with respect to all layers. Note that here C^*/r is the capacity per real degree of freedom. We define C^* this way since the dithers used in the coding scheme are real Bernoulli (1/2) random variables and are not complex. Different SNR realizations, $\mathrm{SNR}_1(r)$ and $\mathrm{SNR}_2(r)$, can be parameterized by a single parameter t as:

$$\begin{aligned} \mathrm{SNR}_1(r) &= e^{\frac{C^*}{r}+t}-1 && (3.1) \\ \mathrm{SNR}_2(r) &= e^{\frac{C^*}{r}-t}-1 && (3.2) \\ |t| &\le \frac{C^*}{r} \end{aligned}$$

Hence we would like the coding scheme to be universal in the sense that it achieves capacity for any channel realization, t, and rateless in the sense that it achieves capacity no matter how many repetition blocks, r, the decoder needs. In other words, here we are addressing both types of uncertainty introduced in Chapter 1.

Note that the case with $r=1$, is the case when the maximum rate in the channel is fixed and the rateless property is not used.

The main tools used in this coding scheme are:

- layering with equal power allocation
- random i.i.d. Bernoulli (1/2) multiplicative dithering

- overlapping and staggering of codewords

- repetition for rateless coding.

The decoder uses a simple MRC receiver along with successive cancellation. Overlapping the codewords generates the desired symmetry across the layers.

In the remainder of the chapter, we describe the encoding and decoding structure of the resulting sub-block structured code. We prove that this code is capacity achieving in the limit of $L \to \infty$ and as long as the number of layers is increased exponentially with C^*. We then perform an approximate efficiency analyses when the number of layers is scaled linearly with C^* and obtain upper and lower bounds on the efficiency. This analysis shows that the efficiency drops at high C^* values if the number of layers are increased only linearly with C^*. We then perform an exact efficiency analysis and show that even though at very large C^*'s the efficiency is low, at most practical ranges, the efficiency is reasonably high using a base code of sufficiently low rate. Finally we briefly examine the use of faster than Nyquist (FTN) signaling to enable the design to perform efficiently at higher C^* values.

■ 3.1 Code Structure

As discussed, layering and repetition alone do not generate the required symmetry that solves the rate and power allocation problems. The idea used in this sub-block structured code is to construct this symmetry by overlapping and staggering of the codewords. This section studies the encoding and decoding structures of the code. As we will show, using layering and overlapping of codewords, combined with successive decoding, generates the required symmetry.

■ 3.1.1 Encoding

To describe the structure of the code, we break it down into its basic units. This basic unit of the code structure is shown in Figure 3-1. It consists of L codewords, where L is the number of layers, overlapped and placed in a diagonal manner on top of one another. Each layer of the basic unit is a single codeword. Within each codeword, we can identify L sub-blocks that are overlapped with different groups of codewords. A block of the code structure consists of a group of M independent basic units. This is shown in Figure 3-2.

We use a common base codebook for all the layers. Later in the chapter we will talk about the specific properties required for this codebook. Each of the L codewords in a basic unit is chosen independently from the common codebook. These L codewords are then overlapped, multiplied, symbol by symbol, by i.i.d. Bernoulli (1/2) dithers, and added on top of each other. Note that each of the L sub-blocks within a codeword sees interference from a different group of codewords. To make the final code block, we generate M independent basic units and group them together (Figure 3-2).

We repeat the same final code block on both sub-channels with different i.i.d. Bernoulli (1/2) dithers. For a rateless code, we also repeat the blocks in time, again with independent dithers.

As we have already seen in the previous chapters for the rateless codes of [3] and for MRC decoding in Chapter 3, the role of i.i.d. Bernoulli (1/2) dithering is to make the interference in the repeated codewords uncorrelated. Hence random dithering reduces the mutual information penalty resulting from the spatial and temporal repetition and allows the MRC receiver to approach capacity at a large number of layers.

The reason for grouping M of the basic units in a single block is to make the rate loss due to sending zeros at the start and end of transmission negligible. Looking at the combined code in Figure 3-2, we observe that ML^2 sub-blocks of information are sent in total. However, in the combined block, at the beginning and the end, there are $1+2+\cdots+(L-1) = L(L-1)/2$ sub-block slots where we don't send any information. This means that we have a rate loss approximately equal to:

$$\frac{2L(L-1)/2}{ML^2} \approx \frac{1}{M} \tag{3.3}$$

where M is the number of basic code units combined together. Hence, to make the rate loss negligible, we should pick M such that, $1/M \ll 1$.

■ 3.1.2 Decoding

We can now describe the decoding structure that will generate the desired symmetry. The decoder first combines the code received in the two sub-channels using an MRC receiver. Here, again, the assumption is that channel information is available at the receiver. For a rateless code, MRC is performed both spatially and temporally. If the decoder needs to collect r copies of the structure for example, it will maximally combine the r repeated versions in the two sub-channels.

The decoder then performs successive decoding and cancellation. It starts decoding from the first codeword in layer 1, treating the other still undecoded codewords that overlap with it as noise. It continues by decoding the first codeword of the other layers in a similar fashion. It then moves on to decoding the second codeword in layer 1 treating the other still undecoded codewords that overlap with it as noise and so on. This decoding order is shown in Figure 3-3. Using this decoding scheme, each of the sub-blocks within a codeword observes a different interference, i.e., the interference varies within a single codeword. However, using successive cancellation, all codewords in all layers see the same pattern of interference, i.e., the first sub-block sees no interference and the last sub-block sees interference from $L-1$ other codewords. This shows that the coding scheme is symmetric with respect to all layers and hence we can assign equal rates and powers to all layers in both sub-channels. The symmetry of this construction therefore solves the rate allocation problem. However, this construction introduces unknown time-variation within a single codeword that complicates the base code design and analysis because of the uncertainty of the SNR pair. If the time-variation was known, we could design a base code for a time-varying scalar Gaussian channel instead of a time-invariant one. However, here the time-variation is not known and the codebook design is not as simple. This will be discussed in more detail in Section 3.5.

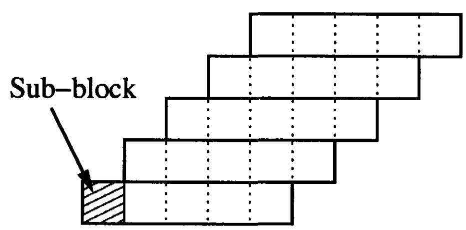


Figure 3-1. Basic unit of the layered code. The basic unit consists of L codewords overlapped and placed diagonally on top of each other. Here, number of layers is 5. Within each codeword, there exist L sub-blocks each with a different interference.

■ 3.2 Capacity-achieving Behavior in the Limit of Large L

Here we look at the behavior of the total achievable rate in the limit of a large number of layers, L, and show that it is capacity achieving in the limit. We will consider the general rateless case. The case where rateless is not required corresponds to $r = 1$. As we increase the number of layers, we hope to be able to achieve a total rate close to capacity, C^*. Let's look at one of the M basic units in a group. The maximum codebook rate for the lth

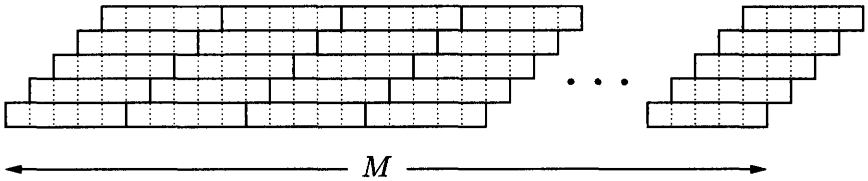


Figure 3-2. A block of the sub-block structured code. Each block consists of M independent basic units, hence it consists of ML sub-blocks. This block is repeated on the two sub-channels with independent Bernoulli (1/2) dithers. For the rateless code, the block is also repeated in time, again with independent dithers.

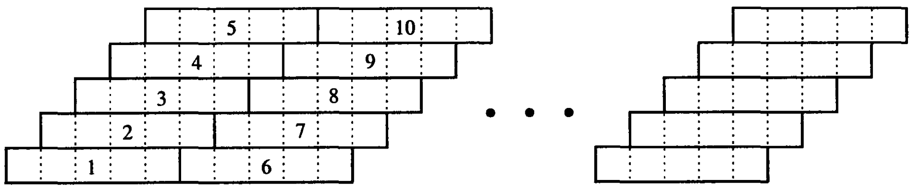


Figure 3-3. Decoding order for the sub-block structure. The decoder first decodes the first codeword of every layer and then moves to the second codeword and continues in this fashion until it finally decodes the Mth and last codeword of every layer. Using this decoding order, every layer sees exactly the same pattern of interference.

layer after collecting r copies and MRC, using a worst-case Gaussian assumption for the interference, is given by:

$$\begin{aligned} I_l &= \frac{1}{2L}\sum_{i=1}^{L}\log\left(1+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_1(r)}}+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_2(r)}}\right) \\ &= \frac{1}{2L}\sum_{i=1}^{L}\log\left(1+\mathrm{SNR}_1(i,r)+\mathrm{SNR}_2(i,r)\right) \end{aligned} \tag{3.4}$$

where,

$$\begin{aligned} \mathrm{SNR}_1(i,r) &= \frac{rP/L}{(i-1)\frac{P}{L}+N_1(r)} \\ &= \frac{r/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_1(r)}} \end{aligned}$$

and $\mathrm{SNR}_2(i,r)$ is defined symmetrically. Note that since all the layers are symmetric, this is true for any arbitrary layer. Hence, the maximum achievable rate in the overall code is given by:

$$
\begin{aligned}
I_{\text{tot}} &= LI_l \\
&= \frac{1}{2}\sum_{i=1}^{L}\log\left(1+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_1(r)}}+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_2(r)}}\right)
\end{aligned} \tag{3.5}
$$

Now if we pick L to be sufficiently large, $\mathrm{SNR}_1(i,r)$ and $\mathrm{SNR}_2(i,r)$ will be arbitrarily small. Using the Taylor series expansion of the log function around 0, we know that for small x:

$$
\log(1+x) \approx x
$$

to a first order approximation. Therefore we can make the first order approximations,

$$
\begin{aligned}
&\log\left(1+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_1(r)}}+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_2(r)}}\right) \\
&\approx \frac{r/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_1(r)}}+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_2(r)}} \\
&\approx \log(1+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_1(r)}})+\log(1+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_2(r)}}) \\
&\approx r\log(1+\frac{1/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_1(r)}})+r\log(1+\frac{1/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_2(r)}})
\end{aligned} \tag{3.6}
$$

In the limit, these approximations become exact. Thus, substituting in (3.5), for the total rate in the limit of $L \to \infty$ we have:

$$
\begin{aligned}
\lim_{L\to\infty} I_{\text{tot}} &= \sum_{i=1}^{L}\frac{r}{2}\log(1+\frac{1/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_1(r)}})+\sum_{i=1}^{L}\frac{r}{2}\log(1+\frac{1/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_2(r)}}) \\
&= \frac{r}{2}\log(1+\mathrm{SNR}_1(r))+\frac{r}{2}\log(1+\mathrm{SNR}_2(r)) \\
&= C^*
\end{aligned} \tag{3.7}
$$

and the scheme is capacity achieving.

Now if L is sufficiently large, we can still use the approximation in (3.6) to say that the code will be approximately capacity achieving, i.e., (3.7) indicates that $I_{\text{tot}} \approx C^*$. It is now important to examine the rate at which we should increase L as a function of C^* for the above approximations to be reasonable and hence for the code to be approximately capacity achieving. We claim that the rate of increase of the number of layers, L, has be exponential with C^*.

Proposition 3.2.1. *For the code to be approximately capacity achieving, L has to grow exponentially with C^*.*

Proof. For the approximation in (3.6) to be reasonable, all the layer SNR's must be sufficiently small. Hence, it is enough to make sure that L is sufficiently large for $\max_{i,r,k} \mathrm{SNR}_k(i,r)$ to be sufficiently small. This maximum SNR is the SNR of the sub-block with no interference ($i = 1$) and in the higher SNR sub-channel. We therefore want:

$$\frac{r/L}{1/\max(\mathrm{SNR}_1(r), \mathrm{SNR}_2(r))} = \frac{r\max(\mathrm{SNR}_1(r), \mathrm{SNR}_2(r))}{L} \to 0 \qquad \forall r$$

.

Maximum sub-channel SNR condition happens in a realization where one of the sub-channels is off, i.e.,

$$\max(\mathrm{SNR}_1(r), \mathrm{SNR}_2(r)) \leq e^{\frac{2C^*}{r}} - 1$$

Hence we must have:

$$\frac{re^{\frac{2C^*}{r}} - r}{L} \to 0 \qquad \forall r$$

If we look at the behavior of the above function with respect to r we have:

$$\frac{\partial}{\partial r}\frac{re^{\frac{2C^*}{r}} - r}{L} = \frac{e^{\frac{2C^*}{r}}(1 - \frac{2C^*}{r}) - 1}{L}$$

Now making a change of variable, $x = C^*/r$, we see that the derivative has the form $e^x(1-x) - 1$. For this function,

$$e^x(1-x) - 1 = 0 \Rightarrow x = 0$$

Hence the derivative approaches zero when $C^*/r \to 0$, i.e., $r \to \infty$, and is negative everywhere else. Therefore the function $(re^{\frac{2C^*}{r}} - r)/L$ is monotonically decreasing with r and takes its maximum at $r = 1$. As a result the condition for L simplifies to:

$$\frac{e^{2C^*} - 1}{L} \to 0 \Rightarrow L \gg e^{2C^*}$$

□

This result implies that in order to get close to capacity, we must increase the number of layers exponentially with C^*. This is not desirable. In practice we would like to pick a base codebook of a reasonably low rate, R, and use that for the coding scheme. This means that we would like to increase the number of layers linearly with C^* and use the same codebook to achieve a good performance. Also encoding and successive decoding of a large number of layers is not easy in practice and as a result, increasing L linearly with C^* is appealing.

We should therefore perform an efficiency analysis for the sub-block structured coding scheme to determine its efficiency performance as a function of C^* when L is scaled linearly with C^* and for every r. This will be done in the next section.

■ 3.3 Efficiency Analysis

In this section we perform an efficiency analysis for the sub-block structured coding scheme to examine its efficiency performance when L is increased linearly with C^* as $C^*/L = R$ for some constant R. We are interested in this performance as a function of C^* and for every r. We will perform both an approximate analysis and an exact one. Using the approximate analysis, we will derive upper and lower bounds on the efficiency and show that they can be tight. The approximate analysis of the efficiency makes a worst case Gaussian assumption on the overall interference of each sub-block. In the exact analysis, the exact distribution of noise is found analytically and the corresponding mutual information and efficiency are calculated numerically in MATLAB.

■ 3.3.1 Approximate Gaussian Analysis

Here we perform an approximate worst-case analysis assuming that the overall interference for each sub-block is Gaussian after combining the repeated copies of the codewords. Our goal is to find upper and lower bounds on efficiency when the number of layers, L, is increasing linearly with C^*.

Upper-bound on Efficiency

We will now find an upper bound on the efficiency of the scheme using a Gaussian base code of arbitrary rate, R^*. Using any base code of arbitrary rate, R^*, we would ideally want our scheme to work for any r (if rateless). Taking the limit as $r \to \infty$, we can calculate an upper bound on the achievable rate per layer, R_{up}, by calculating the achievable rate per layer in the limit and therefore find an upper bound on the efficiency, $R_{\text{up}}L/C^*$ (note that R^* has to be made smaller than R_{up} for the scheme to work).

Proposition 3.3.1. *The total achievable rate, as $r \to \infty$, is given by:*

$$\lim_{r\to\infty} \frac{1}{2L}\sum_{i=1}^{L} \log(1 + \frac{r/L}{\frac{i-1}{L} + \frac{1}{SNR_1(r)}} + \frac{r/L}{\frac{i-1}{L} + \frac{1}{SNR_2(r)}}) = \frac{1}{2}\log(1 + \frac{2C^*}{L}) \tag{3.8}$$

Proof. As we saw in (3.4), for a fixed C^*, the achievable rate per layer using a Gaussian approximation of the total noise is given by:

$$I_l(r) = \frac{1}{2L}\sum_{i=1}^{L} \log(1 + \frac{r/L}{\frac{i-1}{L} + \frac{1}{\text{SNR}_1(r)}} + \frac{r/L}{\frac{i-1}{L} + \frac{1}{\text{SNR}_2(r)}})$$

Let's look at each term of the sum separately in the limit. Using the SNR parameterizations in (3.1) and (3.2), we have:

$$\begin{aligned} &\lim_{r\to\infty} \frac{r/L}{\frac{i-1}{L} + \frac{1}{e^{\frac{C^*}{r}+t}-1}} + \frac{r/L}{\frac{i-1}{L} + \frac{1}{e^{\frac{C^*}{r}-t}-1}} \\ &= \lim_{r\to\infty} \frac{r(e^{\frac{C^*}{r}+t}-1)}{(i-1)(e^{\frac{C^*}{r}+t}-1)+L} + \frac{r(e^{\frac{C^*}{r}-t}-1)}{(i-1)(e^{\frac{C^*}{r}-t}-1)+L} \end{aligned} \tag{3.9}$$

Parameter t can take values in the range $|t| \leq C^*/r$. Hence $r \to \infty \Rightarrow t \to 0$ and therefore:

$$\lim_{r\to\infty} (e^{\frac{C^*}{r}\pm t} - 1) = 0 \tag{3.10}$$

Using (3.10), (3.9) reduces to:

$$\begin{aligned} &\lim_{r\to\infty} \frac{r(e^{\frac{C^*}{r}+t}-1) + r(e^{\frac{C^*}{r}-t}-1)}{L} \\ &= \lim_{r\to\infty} \frac{1}{L}\left[(C^*+tr) + \frac{(C^*+tr)^2}{2r} + \cdots + (C^*-tr) + \frac{(C^*-tr)^2}{2r} + \cdots\right] \\ &= \frac{2C^*}{L} + \frac{1}{L}\sum_{i=2}^{\infty} \frac{(C^*+tr)^i}{i!r^{i-1}} + \frac{(C^*-tr)^i}{i!r^{i-1}} \end{aligned} \tag{3.11}$$

where we used the Taylor series expansion of the exponential function at 0. But $|t| \leq C^*/r$, hence:

$$(C^*+tr)^n + (C^*-tr)^n \leq 2(2C^*)^n$$

and:

$$\begin{aligned}\sum_{i=2}^{\infty}\frac{(C^*+tr)^i}{i!r^{i-1}}+\frac{(C^*-tr)^i}{i!r^{i-1}} &\leq 4C^*\sum_{i=1}^{\infty}(\frac{2C^*}{r})^i \\ &= 4C^*(\frac{1}{1-\frac{2C^*}{r}}-1)\end{aligned}$$

Here we used the fact that the above expression is a geometric series and since $r \to \infty$, for any C^*, as long as we take $r > 2C^*$, the terms in the geometric series are less than 1 in magnitude. Now taking the limit as $r \to \infty$ we have:

$$\lim_{r\to\infty} 4C^*(\frac{1}{1-\frac{2C^*}{r}}-1) = 0$$

Since the summation terms, $C^* \pm tr \geq 0$ in (3.11), we have:

$$\lim_{r\to\infty}\sum_{i=2}^{\infty}\frac{(C^*+tr)^i}{i!r^{i-1}}+\frac{(C^*-tr)^i}{i!r^{i-1}} = 0$$

Finally, substituting back into (3.11) and (3.9), we get:

$$\lim_{r\to\infty}\frac{r/L}{\frac{i-1}{L}+\frac{1}{e^{\frac{C^*}{r}+t}-1}}+\frac{r/L}{\frac{i-1}{L}+\frac{1}{e^{\frac{C^*}{r}-t}-1}} = \frac{2C^*}{L} \tag{3.12}$$

And,

$$\lim_{r\to\infty}\frac{1}{2L}\sum_{i=1}^{L}\log(1+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_1(r)}}+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_2(r)}}) = \frac{1}{2L}L\log(1+\frac{2C^*}{L}) = \frac{1}{2}\log(1+\frac{2C^*}{L})$$

□

We can now find an upper bound on the efficiency of any base code of rate R^*. The efficiency for a code of rate R^* is given by,

$$\eta(R^*) = \frac{R^*L}{C^*} \tag{3.13}$$

To find an upper bound on efficiency, note that the rateless coding scheme should work for every r and hence it should work as $r \to \infty$. Therefore for a rate R^* to be achievable in every layer and for every r it has to satisfy:

$$R^* \leq \frac{1}{2}\log(1 + \frac{2C^*}{L})$$

i.e.,

$$L \leq \frac{2C^*}{\exp(2R^*) - 1} \tag{3.14}$$

Substituting (3.14) into (3.13), we get:

$$\eta(R^*) \leq \frac{2R^*}{\exp(2R^*) - 1} \tag{3.15}$$

For example for a rate of $R^* = 1/6$ bits, the upper bound on achievable efficiency is 0.8889. We may argue that this might not be a true upper bound since we have used a worst-case Gaussian approximation of the noise. However, as we will see in the exact noise and efficiency analysis, as we increase r, the Gaussian approximation becomes very close to the true value and hence the upper-bound (3.15) is valid. Note that this upper bound is for the rateless code design.

Lower-bound on Efficiency

Here we will find a lower bound on the efficiency as C^* increases and L linearly with it. Recall:

$$C(r) = \frac{1}{2}\log(1 + \mathrm{SNR}_1(r)) + \frac{1}{2}\log(1 + \mathrm{SNR}_2(r)) = \frac{C^*}{r} \tag{3.16}$$

If an overall channel has a capacity $C(r)$, the decoder collects r copies of the same codeword to accumulate a rate close to C^*. As C^* increases, the number of layers is scaled linearly with it to always keep $R = C^*/L$ where R is some fixed constant. Here we examine the behavior of the maximum achievable rate per layer as C^* increases to find a lower bound on efficiency. We will show that if L is increased only linearly with C^*, the lower bound on efficiency for the rateless code is 0 and this lower bound becomes tight as $C^* \to \infty$. If we are not interested in the rateless property, the lower bound is 1/2 and will be tight as

$C^* \to \infty$. The reason for this, as we will show next, is that in the limit of large C^* the first sub-block (the one with no interference) dominates the mutual information and is no longer in the low SNR regime. Hence repetition incurs a mutual information penalty and the efficiency drops.

To prove these results we will first show a couple of inequalities on the rate contribution of different sub-blocks and then prove a couple of claims and finally arrive at the bounds. Recall that the achievable rate per layer I_l as a function of r and t is given by (3.4):

$$\begin{aligned} I_l(r,t) &= \frac{1}{2L}\sum_{i=1}^{L}\log(1+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_1(r)}}+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\mathrm{SNR}_2(r)}}) \\ &= \frac{1}{2L}\sum_{i=1}^{L}\log(1+\frac{r/L}{\frac{i-1}{L}+\frac{1}{e^{\frac{C^*}{r}-t}-1}}+\frac{r/L}{\frac{i-1}{L}+\frac{1}{e^{\frac{C^*}{r}+t}-1}}) \end{aligned}$$

Hence the maximum achievable rate per layer for the scheme is given by:

$$R_{\max} = \min_{r,t} I_l(r,t) \tag{3.17}$$

We can split the expression for $I_l(r,t)$ into two parts:

$$\begin{aligned} I_l(r,t) = &\frac{1}{2L}\log(1+\frac{r}{L}(e^{\frac{C^*}{r}-t}-1)+\frac{r}{L}(e^{\frac{C^*}{r}+t}-1)) \\ &+\frac{1}{2L}\sum_{i=2}^{L}\log(1+\frac{r/L}{\frac{i-1}{L}+\frac{1}{e^{\frac{C^*}{r}-t}-1}}+\frac{r/L}{\frac{i-1}{L}+\frac{1}{e^{\frac{C^*}{r}+t}-1}}) \end{aligned}$$

The first part corresponds to the rate contribution of the first sub-block with no interference and the second part corresponds to the rate contribution of all other sub-blocks. For the first term we have:

$$\begin{aligned} \frac{1}{2L}\log(1+\frac{r}{L}(e^{\frac{C^*}{r}-t}-1)+\frac{r}{L}(e^{\frac{C^*}{r}+t}-1)) &= \frac{1}{2L}\log(1+\frac{r}{L}e^{\frac{C^*}{r}}(e^{t}+e^{-t})-\frac{2r}{L}) \\ &\geq \frac{1}{2L}\log(1+\frac{2r}{L}e^{\frac{C^*}{r}}-\frac{2r}{L}) \end{aligned} \tag{3.18}$$

Now since $|t| \leq C^*/r$ and therefore $e^{\frac{C^*}{r}\pm t}-1 \geq 0$, the rate contribution of the other sub-blocks satisfies:

$$\begin{aligned}
\frac{1}{2L}\sum_{i=2}^{L}\log(1+\frac{r/L}{\frac{i-1}{L}+\frac{1}{e^{\frac{C^*}{r}-t}-1}}+\frac{r/L}{\frac{i-1}{L}+\frac{1}{e^{\frac{C^*}{r}+t}-1}}) &\leq \frac{1}{2L}\sum_{i=2}^{L}\log(1+\frac{2r/L}{(i-1)/L}) \\
&= \frac{1}{2L}\sum_{i=2}^{L}\log(1+\frac{2r}{i-1}) \qquad (3.19)
\end{aligned}$$

To get the desired lower bound, we will now prove a couple of claims and use them in deriving the bounds.

Proposition 3.3.2. *When increasing L linearly with C^* as: $R = C^*/L$ for some constant R, and in the limit of large C^*, the mutual information contributed by the first sub-block dominates the sum in* (3.4) *for the achievable rate per layer no matter what the channel realization is, i.e.,*

$$\begin{aligned}
\lim_{C^*\to\infty} I_l(r,t) &= \lim_{C^*\to\infty}\frac{1}{2L}\sum_{i=1}^{L}\log(1+\frac{r/L}{\frac{i-1}{L}+\frac{1}{e^{\frac{C^*}{r}-t}-1}}+\frac{r/L}{\frac{i-1}{L}+\frac{1}{e^{\frac{C^*}{r}+t}-1}}) \\
&= \lim_{C^*\to\infty}\frac{1}{2L}\log(1+\frac{r}{L}(e^{\frac{C^*}{r}-t}-1)+\frac{r}{L}(e^{\frac{C^*}{r}+t}-1)) \\
&= \frac{R}{2r} \qquad (3.20)
\end{aligned}$$

Proof. Using (3.18) and (3.19), we construct the sum,

$$\frac{1}{2L}\log(1+\frac{2r}{L}e^{\frac{C^*}{r}}-\frac{2r}{L})+\frac{1}{2L}\sum_{i=2}^{L}\log(1+\frac{2r}{i-1})$$

and take its limit as $C^*\to\infty$ to show that this limit is the same as the limit of its first term, i.e., the limit of the second term is zero. Note that from (3.18) and (3.19), the first term of this expression is the lower bound of the first sub-block rate for any channel realization and the summation in this expression is the upper bound of the rate contribution of all other sub-blocks for any channel realization. Hence if the above result holds, we can conclude that regardless of channel realization, the rate of the first sub-block dominates the total rate in the limit of large C^*.

Knowing $C^* = RL$ and for sufficiently large L, the first term becomes:

$$
\begin{aligned}
\frac{1}{2L}\log(1+\frac{2r}{L}e^{\frac{RL}{r}}-\frac{2r}{L}) &\approx \frac{1}{2L}\log(1+\frac{2r}{L}e^{\frac{RL}{r}}) \\
&\approx \frac{1}{2L}\log(\frac{2r}{L}e^{\frac{RL}{r}}) \\
&= \frac{\log(2r)}{2L} - \frac{\log(L)}{2L} + \frac{1}{2L}\frac{RL}{r}
\end{aligned}
$$

Now taking the limit of this term as $L \to \infty$ (equivalent to $C^* \to \infty$) we have:

$$
\begin{aligned}
\lim_{L\to\infty}\frac{1}{2L}\log(1+\frac{2r}{L}e^{\frac{RL}{r}}-\frac{2r}{L}) &= \lim_{L\to\infty}\left(\frac{\log(2r)}{2L} - \frac{\log(L)}{2L} + \frac{1}{2L}\frac{RL}{r}\right) \\
&= \frac{R}{2r} = \frac{C^*}{2rL} \qquad (3.21)
\end{aligned}
$$

We can approximate the overall sum for large L by:

$$
\begin{aligned}
&\frac{1}{2L}\log(1+\frac{2r}{L}e^{\frac{RL}{r}}-\frac{2r}{L}) + \frac{1}{2L}\sum_{i=2}^{L}\log(1+\frac{2r}{i-1}) \\
&\approx \frac{\log(2r)}{2L} - \frac{\log(L)}{2L} + \frac{1}{2L}\frac{RL}{r} + \frac{1}{2L}\sum_{i=1}^{L-1}\log(1+\frac{2r}{i}) \\
&= \frac{\log(2r)}{2L} - \frac{\log(L)}{2L} + \frac{R}{2rL} + \frac{1}{2L}\sum_{i=1}^{L-1}\left[\log(1+\frac{2r}{i}) + \frac{R}{r}\right]
\end{aligned}
$$

Now taking the limit as $L \to \infty$:

$$
\begin{aligned}
&\lim_{L\to\infty}\left(\frac{\log(2r)}{2L} - \frac{\log(L)}{2L} + \frac{R}{2rL} + \frac{1}{2L}\sum_{i=1}^{L-1}\left[\log(1+\frac{2r}{i}) + \frac{R}{r}\right]\right) \\
&= \lim_{L\to\infty}\frac{1}{2L}\sum_{i=1}^{L-1}\left[\log(1+\frac{2r}{i}) + \frac{R}{r}\right]
\end{aligned}
$$

For any $\epsilon > 0$ small, we can pick an L_o large enough such that $\log(1 + (2r)/L_o) + R/r \leq$

$\epsilon + R/r$. Hence:

$$\begin{aligned}
&\lim_{L\to\infty} \frac{1}{2L} \sum_{i=1}^{L-1} \left[\log(1+\frac{2r}{i}) + \frac{R}{r}\right] \\
&\leq \lim_{L\to\infty} \frac{1}{2L} \sum_{i=1}^{L_o-1} \left[\log(1+\frac{2r}{i}) + \frac{R}{r}\right] + \lim_{L\to\infty} \frac{1}{2L} \sum_{i=L_0}^{L-1} \left[\epsilon + \frac{R}{r}\right] \\
&= \lim_{L\to\infty} \frac{M}{2L} + \lim_{L\to\infty} \frac{1}{2L}(L-L_o)(\epsilon + \frac{R}{r}) \\
&= \frac{R}{2r} + \frac{\epsilon}{2}
\end{aligned}$$

where $M = \sum_{i=1}^{L_o-1} [\log(1+2r/i) + R/r]$ is a finite constant. Since this can be done for any $\epsilon > 0$, we conclude:

$$\lim_{L\to\infty} \frac{1}{2L} \sum_{i=1}^{L-1} \left[\log(1+\frac{2r}{i}) + \frac{R}{r}\right] = \frac{R}{2r} = \frac{C^*}{2rL} \tag{3.22}$$

which is the same as the limit of the first term in (3.21). Hence (3.20) holds. □

Hence regardless of the channel realization, as $C^* = RL$ becomes large, the first term — namely the rate contribution of the sub-block with no interference — dominates the achievable rate per layer.

Proposition 3.3.3. *For a rateless code that allows the rateless property up to r number of repetitions, when $C^* \to \infty$, the worst case achievable rate per layer happens at the channel realization with $t = 0$ and the efficiency of the scheme is given by:*

$$\eta = \frac{1}{2r} \tag{3.23}$$

Proof. From (3.18), the worst case of the dominating mutual information contribution, that of the first sub-block, is when $t = 0$ or $\mathrm{SNR}_1(r) = \mathrm{SNR}_2(r)$. Now using (3.17), the worst case achievable rate per layer when $C^* \to \infty$ happens at this channel realization for maximum number of repetitions, r, and therefore the efficiency of the scheme is given by:

$$\eta(r) = \frac{LR_{\max}}{C^*} = \frac{(LC^*)/(2rL)}{C^*} = \frac{1}{2r}$$

□

This basically says that in a rateless code serving users with capacities $C(r)$ up to some r, we have to pick a rate of $C^*/2Lr$ per layer instead of C^*/L and hence the efficiency is $1/2r$.

Having proved the above two results we can find the desired lower bounds. If we require the code to be completely rateless, it must work for every r. Taking the limit of (3.23) as $r \to \infty$, we get:

$$\lim_{r\to\infty} \eta(r) = \lim_{r\to\infty} \frac{1}{2r} = 0 \tag{3.24}$$

This immediately tells us that the lower bound on efficiency using the Gaussian approximation of noise is 0 since we want the scheme to work for every r. Note that this bound is tight at $C^* \to \infty$. The reason is that at the limit of $C^* \to \infty$, the number of layers also tends to infinity since $L = C^*/R$ and R is a fixed constant. Having many layers means that the interference is the sum of an infinite number of independent layers. Thus, according to the central limit theorem, the distribution of the interference approaches that of a Gaussian distribution with the same variance, which was the assumption on the interference in deriving the bound in (3.24).

If the code is not required to be rateless then the minimum efficiency is given by substituting $r = 1$ in (3.23),

$$\eta = \frac{1}{2} \qquad r = 1 \tag{3.25}$$

which again is tight at $C^* \to \infty$.

The reason for this low efficiency performance at large values of C^* is that within each codeword, there exists a sub-block that experiences no interference from other layers. Now if the Gaussian white noise is very small, this sub-block will dominate in terms of mutual information contributed to the total mutual information and will not be at a low SNR regime (since we are growing L only linearly with C^*). This means that repetition in both time and space incurs a mutual information penalty. Had we chosen L to change exponentially with C^* we would not have run into this problem. As a result, unlike the rateless coding scheme in [3], there is no non-zero lower-bound on efficiency that holds for every r and C^* for a given ratio of C^*/L.

Having examined the efficiency performance at the large values of C^*, we should now perform an exact efficiency analysis to examine this performance for other C^* values. Specifically, we are interested in this performance in the range of C^* values that are usually dealt with in practice. The following section will perform this exact analysis.

3.3.2 Exact Noise Analysis

This section presents an exact noise and efficiency analysis of the sub-block structured coding scheme. As discussed previously, in this scheme, different symbols in the same block experience different interferences. The structure of the noise coming from each layer is a Gaussian mixture for odd repetitions and a Gaussian mixture plus a discrete component at 0 for even repetitions. Let $\bar{\mathbf{x}}_j$ be taken from an i.i.d. Gaussian codebook with unit variance. The codeword in the jth layer is given by $\mathbf{x}_j = 1/\sqrt{L} \cdot \bar{\mathbf{x}}_j$ ($1/L$ is the power per layer). If we look at the repetition versions of the same layer j, each is dithered by an independent Bernoulli (1/2) sequence, $\mathbf{d}_j(r)$, i.e., for the jth layer copies we have:

$$\mathbf{x}_j \odot \mathbf{d}_j(1), \cdots, \mathbf{x}_j \odot \mathbf{d}_j(r)$$

The ith received block is $\mathbf{y}(i) = \mathbf{x}(i) + \mathbf{z}(i)$, where $\mathbf{x}(i)$ is the code block in Figure 3-2 and $\mathbf{z}(i)$ is the white Gaussian noise vector with variance N. Let us look at a sub-block in this block. It consists of the summation of L sub-blocks with different interferences. Let's look at the component sub-block having h interfering layers. The MRC performs the following averaging for a sub-block having h interfering layers (and therefore $L-h-1$ non-interfering layers) in an arbitrary layer l (all layers symmetric):

$$\mathbf{y}_l = \sum_{i=1}^{r} \frac{1}{r} \cdot \mathbf{d}_l(i) \odot \frac{\mathbf{y}(i) - \sum_{k \in L-h-1 \text{ non-interf-layers}} \frac{1}{\sqrt{L}} \hat{\bar{\mathbf{x}}}_k \odot \mathbf{d}_k(i)}{1/\sqrt{L}}$$

where all the vectors involved contain the symbols in that sub-block only. Here $\hat{\bar{\mathbf{x}}}_k$ is the result of decoding the kth layer corresponding sub-block. Assuming perfect decoding of non-interfering layers, i.e. $\hat{\mathbf{x}}_k = \mathbf{x}_k$, we have:

$$\mathbf{y}_l = \bar{\mathbf{x}}_l + \sum_{i=1}^{r} \frac{1}{r} \left(\frac{1}{1/\sqrt{L}} \left(\sum_{k \in h \text{ interf-layers}} \mathbf{x}_k \odot \mathbf{d}_l(i) \odot \mathbf{d}_k(i) + \mathbf{z}(i) \right) \right)$$

We can write this as:

$$\mathbf{y}_l = \bar{\mathbf{x}}_l + \mathbf{v}_l + \tilde{\mathbf{z}}_l$$

where,

$$\tilde{\mathbf{z}}_l = \sum_{i=1}^{r} \frac{\sqrt{L}}{r} \mathbf{z}(i)$$

and the total interference, $\mathbf{v}_l$, is of the form:

$$\mathbf{v}_l = \sum_{i=1}^{r} \frac{1}{r/\sqrt{L}} \left(\sum_{k \in h \text{ interf-layers}} \mathbf{x}_k \odot \mathbf{d}_l(i) \odot \mathbf{d}_k(i) \right)$$

Note that the product of two independent Bernoulli (1/2) random variables is itself Bernoulli (1/2) and hence $\mathbf{d}_l(i) \odot \mathbf{d}_k(i)$ is again a Bernoulli (1/2) sequence. For simplicity we redefine this new sequence as $\mathbf{d}_k$ for the purpose of finding the distributions. We can write the total interference as $\mathbf{v}_l = \sum_{j \in h \text{ interf-layers}} \mathbf{w}_j$, where $\mathbf{w}_j$ is the interference contributed by the jth interfering layer:

$$\mathbf{w}_j = \sum_{i=1}^{r} \frac{1}{r/\sqrt{L}} \mathbf{x}_j \odot \mathbf{d}_j(i) = \sum_{i=1}^{r} \frac{1}{r} \bar{\mathbf{x}}_j \odot \mathbf{d}_j(i)$$

The Gaussian noise contribution, $\tilde{\mathbf{z}}_l$, has a variance of $(NL)/r$ after MRC, i.e.,

$$\tilde{\mathbf{z}}_l \sim N(0, \frac{NL}{r})$$

Let's look at one symbol of $\bar{\mathbf{x}}_j$, say the mth symbol $\bar{\mathbf{x}}_j(m)$ and the corresponding r dither terms $\mathbf{d}_j(1,m), \cdots, \mathbf{d}_j(r,m)$. The sum of these dither terms can be an odd number when r is odd and either an even number or 0 if r is even. Let's assume that r is odd, i.e., $r = 2k+1$, for some k. Let's assume that the difference between the number of 1's and -1's in the sequence $\mathbf{d}_j(1,m), \cdots, \mathbf{d}_j(r,m)$ is $d^+ - d^- = 2k' + 1$. We also know that $r = d^+ + d^-$. From these two equations:

$$d^+ = k' + k + 1$$

Therefore for the distribution of the layer noise we have:

$$f_w(w) = \sum_{k'=-(k+1)}^{k} \frac{\binom{2k+1}{k'+k+1}}{2^r} N(0, (\frac{2k'+1}{r})^2) \qquad r = 2k+1$$

where $N(m, \sigma^2)$, is the Gaussian distribution with mean m and variance σ^2.

For even $r = 2k$, let's assume that $d^+ - d^- = 2k'$. We also know that $r = d^+ + d^-$. Hence, $d^+ = k + k'$. Also if the sum of the terms is zero, then $d^+ = d^- = k$. Therefore for the noise distribution coming from an interfering layer we have:

$$f_w(w) = \sum_{k'=-k, k' \neq 0}^{k} \frac{\binom{2k}{k+k'}}{2^r} N(0, (\frac{2k'}{r})^2) + \frac{\binom{2k}{k}}{2^r} \delta(w) \qquad r = 2k$$

Now for a sub-block that has l interfering layers, the interference will have the following distribution:

$$f_v(v) = \text{conv}_l(f_w(v))$$

where $\text{conv}_l(y)$ denotes the l-fold convolution of y with itself.

3.3.3 Numerical Results on Exact Efficiency Analysis

Having the exact noise distributions found in the previous section, numerical analyses are performed to find the exact efficiency of the sub-block structured rateless scheme for $R = C^*/L = 1/2$ bit and $R = C^*/L = 1/6$ bit. In our analysis, C^* is increased and L linearly with it to keep C^*/L constant at R. The number of repetitions is changed from $r = 1$ to $r = 6$. In each case the total noise distribution is found for each of the L sub-block components of a single codeword and from that the entropy of the noise for the corresponding sub-block is calculated. The input is taken from a unit variance Gaussian distribution (since $\bar{\mathbf{x}}_k$ is Gaussian and unit variance.). The output distribution is found by convolving the input and total noise distributions and from that the output entropy. The average mutual information per symbol in the lth sub-block is then calculated as:

$$I(l) = h(y_l) - h(x_l)$$

Here x_l and y_l denote an input and output symbol in the lth sub-block seeing interference from $l-1$ layers and $I(l)$ is the mutual information per symbol of the corresponding sub-

Table 3.1. Efficiencies for $C^*/L = 1/2$ bit with equal power allocation and with one sub-channel off.

C^*	1 bit	2 bit	3 bit	6 bit	12 bit
L	2	4	6	12	24
$r = 2$	0.8709	0.8688	0.8619	0.8295	0.8077
$r = 2$ Approx	0.8685	0.8601	0.8468	0.7892	0.6932
$r = 3$	0.8314	0.8266	0.8191	0.7918	0.6937
$r = 3$ Approx	0.8305	0.8257	0.8178	0.7784	0.6733
$r = 4$	0.8126	0.8096	0.8048	0.7799	0.7100
$r = 4$ Approx	0.8122	0.8093	0.8044	0.7788	0.6953
$r = 5$	0.8017	0.7996	0.7964	0.7790	0.7249
$r = 5$ Approx	0.8015	0.7995	0.7962	0.7787	0.7163
$r = 6$	0.7945	0.7930	0.7907	0.7781	0.7355
$r = 6$ Approx	0.7944	0.7930	0.7906	0.7779	0.7310

block. Finally the average mutual information per symbol is given by the average:

$$I_{\text{avg}} = \frac{1}{L}\sum_{i=1}^{L} I(l)$$

Numerical analyses are performed for the two extreme cases with $N_1 = N_2$ and with one sub-channel off, since the worst case efficiency happens at one of these extreme cases. Tables 3.1, 3.2, 3.3, and 3.4 summarize these results where the approximate efficiencies are calculated assuming a Gaussian distribution of the total noise. Using (3.5), this approximate efficiency is given by:

$$\eta_{\text{approx}} = \frac{I_{\text{tot}}}{C^*} = \frac{1}{2C^*}\sum_{i=1}^{L}\log\left(1 + \frac{r/L}{\frac{i-1}{L} + \frac{1}{\text{SNR}_1(r)}} + \frac{r/L}{\frac{i-1}{L} + \frac{1}{\text{SNR}_2(r)}}\right)$$

Observations

Here we will summarize the numerical results. We observe that for two cases, $r = 1$ and $t = 0$ ($N_1 = N_2$) and $r = 2$ and $t = \pm C^*/r$ (one sub-channel off), the efficiencies are always high and the Gaussian approximation is not close to the true value. However as the number of repetitions, r, increases, the exact efficiency values become close to the approximate values and the noise distributions become smoother and more Gaussian like.

Another important observation is the effect of increasing C^*. As C^* increases and L linearly with it, the efficiency drops. For example performing the numerical analysis at

Table 3.2. Efficiencies for $C^*/L = 1/2$ bit with equal power allocation and $N_1 = N_2$.

C^*	1 bit	2 bit	3 bit	6 bit	12 bit
L	2	4	6	12	24
$r = 1$	0.8709	0.8688	0.8619	0.8295	0.8077
$r = 1$ Approx	0.8685	0.8601	0.8468	0.7892	0.6932
$r = 2$	0.8126	0.8096	0.8048	0.7799	0.7100
$r = 2$ Approx	0.8122	0.8093	0.8044	0.7788	0.6953
$r = 3$	0.7945	0.7930	0.7907	0.7781	0.7355
$r = 3$ Approx	0.7944	0.7930	0.7906	0.7779	0.7310
$r = 4$	0.7856	0.7848	0.7834	0.7760	0.7475
$r = 4$ Approx	0.7856	0.7848	0.7834	0.7760	0.7474
$r = 5$	0.7804	0.7798	0.7789	0.7741	0.7551
$r = 5$ Approx	0.7804	0.7798	0.7789	0.7741	0.7551
$r = 6$	0.7769	0.7765	0.7759	0.7725	0.7590
$r = 6$ Approx	0.7769	0.7765	0.7759	0.7725	0.7590

$C^* = 22$ bits, $L = 44$, and $r = 6$, we observe that the efficiency is as low as 0.6396 and the approximate efficiency is 0.6289. This is consistent with the prediction in our approximate lower bound analysis which suggested that for an arbitrary ratio $C^*/L = R$, there exists C^* large enough for which the efficiency is low and goes down as $1/r$ in the limit of large C^*.

The other observation is that there is no clear pattern in the change of efficiency with r for a fixed C^*. This is probably a result of the fact that the sum in (3.4) does not have a simple behavior with r alone and its behavior with r depends on other parameters such as L and C^* as well.

Figures 3-4 and 3-5 illustrate the total noise distribution for $C^* = 12$ bits and $L = 24$, for $r = 2$ and $r = 5$ and for the realization with one sub-channel off. They also show the Gaussian distribution of the same variance for comparison. We see that in the case of even r's, especially for $r = 2$, the noise distribution has a sharp peak at 0 because of the discrete component in the interference; however, this effect becomes less dominant by increasing r. As r increases, the exact noise distributions become very close to a Gaussian. Also, as we have more interfering layers, the noise distribution becomes more Gaussian as expected.

From comparison of results for $R = 1/2$ bit and $R = 1/6$ bit, we see that the efficiency increases by using more layers in the scheme and hence a lower rate base code as expected. Let's look at the efficiencies for the case with more layers ($R = 1/6$). From Tables 3.3 and 3.4 we see that the efficiency numbers for both extreme cases of SNR pairs and for C^*'s up to 5 bits are all above 90%. For a C^* of 6 bits, the worst efficiency number observed

Table 3.3. Efficiencies for $C^*/L = 1/6$ bit with equal power allocation for the case with one sub-channel off and for different C^*'s.

C^*	1 bit	2 bits	3 bits	4 bits	5 bits	6 bits
L	6	12	18	24	30	36
$r = 2$	0.9466	0.9415	0.9328	0.9239	0.9108	0.8958
$r = 2$ Approx	0.9465	0.9414	0.9324	0.9194	0.9022	0.8816
$r = 3$	0.9307	0.9278	0.9229	0.9158	0.9065	0.8948
$r = 3$ Approx	0.9307	0.9278	0.9228	0.9157	0.9063	0.8944
$r = 4$	0.9229	0.9211	0.9181	0.9137	0.9081	0.9010
$r = 4$ Approx	0.9229	0.9211	0.9181	0.9137	0.9080	0.9010
$r = 5$	0.9183	0.9171	0.9150	0.9121	0.9083	0.9037
$r = 5$ Approx	0.9183	0.9171	0.9150	0.9121	0.9083	0.9037
$r = 6$	0.9152	0.9143	0.9129	0.9108	0.9081	0.9048
$r = 6$ Approx	0.9152	0.9143	0.9129	0.9108	0.9081	0.9048

is 88.16%. Also, whether the case with $N_1 = N_2$ performs better or the case with one sub-channel off, depends on r and C^*. In practice, C^* does not usually exceed 5 bits. This means that the sub-block structured scheme performs reasonably well for practical ranges of C^*. Note that if we are not interested in rateless, i.e., $r = 1$, at the extreme case with $t = \pm C^*/r$, we are capacity achieving and the efficiency at the extreme case with $t = 0$, is again always above 90% for up to $C^* = 5$ bits.

■ 3.3.4 Efficiency Comparison with the Rateless Scheme with Non-uniform Power Allocation Over Scalar Channels

In this section we examine the advantage of time-varying non-uniform power allocation used in [3] for an scalar channel over a uniform power allocation for a scalar channel that we are forced to use in the parallel Gaussian channel because of the uncertainty of the realization. This comparison is done between the performance of these rateless codes over a scalar channel. As was mentioned in Chapter 1, in [3] a time-varying and non-uniform power allocation is used across the layers to equalize the rates. Using this power allocation, the rateless scheme is shown to have a non-zero lower bound on efficiency regardless of how large C^* is when L is increased linearly with C^*. This is in contrast with our efficiency results which indicate that the efficiency goes to 0 as C^* gets large even for the realization with one sub-channel off which is equivalent to a scalar channel. This may seem surprising since here again we are increasing the number of layers linearly with C^* and the channel

Table 3.4. Efficiencies for $C^*/L = 1/6$ bit with equal power allocation for the case of $N_1 = N_2$ and for different C^*'s.

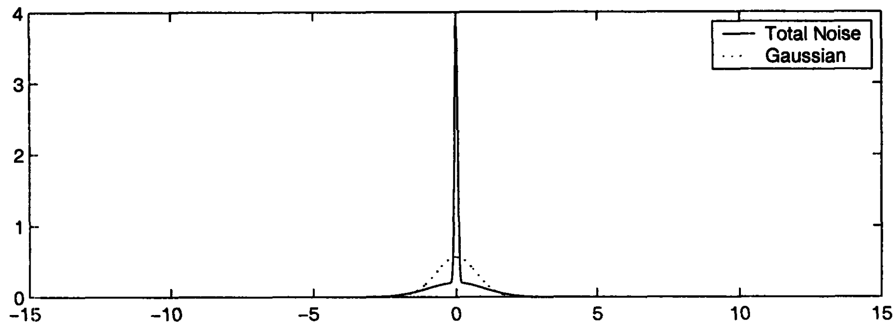
C^*	1 bit	2 bits	3 bits	4 bits	5 bits	6 bits
L	6	12	18	24	30	36
$r = 1$	0.9466	0.9415	0.9328	0.9239	0.9108	0.8958
$r = 1$ Approx	0.9465	0.9414	0.9324	0.9194	0.9022	0.8816
$r = 2$	0.9229	0.9211	0.9181	0.9137	0.9081	0.9010
$r = 2$ Approx	0.9229	0.9211	0.9181	0.9137	0.9080	0.9010
$r = 3$	0.9152	0.9143	0.9129	0.9108	0.9081	0.9048
$r = 3$ Approx	0.9152	0.9143	0.9128	0.9108	0.9081	0.9048
$r = 4$	0.9113	0.9108	0.9100	0.9087	0.9072	0.9053
$r = 4$ Approx	0.9113	0.9108	0.9100	0.9087	0.9072	0.9053
$r = 5$	0.9090	0.9087	0.9081	0.9073	0.9063	0.9051
$r = 5$ Approx	0.9090	0.9087	0.9081	0.9073	0.9063	0.9051
$r = 6$	0.9075	0.9072	0.9068	0.9063	0.9055	0.9047
$r = 6$ Approx	0.9075	0.9072	0.9068	0.9063	0.9055	0.9047

realization with one sub-channel off is exactly equivalent to the scalar channel for which the rateless code in [3] is designed.

To see the reason why in the non-uniform power allocation case increasing the number of layers linearly with C^* is enough to always have a lower bound on efficiency which is bounded away from zero (which is not the case in the parallel channel problem), we will perform an exact noise analysis for this scheme and compare it to the performance of our rateless code over a scalar channel. The comparison is basically between the performance of the rateless scheme used in [3] for a scalar channel and the performance of the rateless scheme in our design for a scalar channel which corresponds to the one sub-channel off realization of parallel channel.

Note that the efficiency performance of our rateless coding scheme over a scalar channel will be exactly the same as the efficiency performance of the rateless coding scheme in [3] if we use uniform time-invariant power allocation over the layers. Meaning, if we remove the staggering from our design and use it over the scalar channel it will have the same efficiency performance. Off course in that case the base code sees a block constant SNR instead of a time-varying one and the base codes and their performance will be different but for the purpose of this comparison we assume to have perfect base codes for both cases.

The exact noise analysis is performed for the rateless scheme with non-uniform power allocation of [3]. The results are shown in Table 3.5. Exact calculations for some of the cases



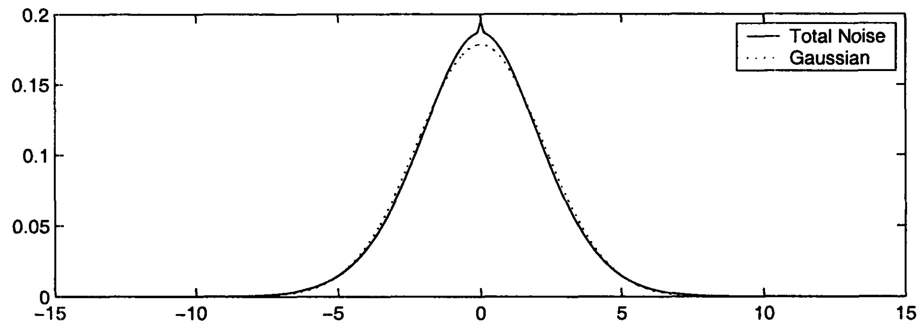


Figure 3-4. Total noise distribution for $C^* = 12$ bits, $L = 24$, and $r = 2$ for a symbol with 1 interfering layer (top) and a symbol with many interfering layers (bottom) when one sub-channel is off. The distribution for $r = 1$ (the non-rateless case) and when $N_1 = N_2$ is the same.

are skipped because of high computational load and closeness to Gaussian approximation. We can see that the approximate efficiency is very close to the exact efficiency in the non-uniform power allocation case. The exact interference is a mixture of 2^r equiprobable Gaussian distributions with different variances determined by the specific power allocation. The total noise is very close to a Gaussian.

Numerical results of the uniform power allocation performance are shown in Table 3.3.

Numerical results show that in the layering scheme with non-uniform power allocation, as r increases, the efficiency drops for a fixed C^*. Also there is no noticeable drop in efficiency as C^* increases which suggests that increasing L linearly with C^* is good enough, in contrast with the behavior of the sub-block structured coding scheme.

The most important observation is that compared to the equal power case, the efficiencies in the non-uniform power allocation case are higher. Even comparing the approximate efficiencies, we see that this is the case. This again confirms that in terms of the number of layers needed, layering with non-uniform power allocation needs fewer layers than the case with equal power allocated to all layers for a scalar channel. One reason for this difference

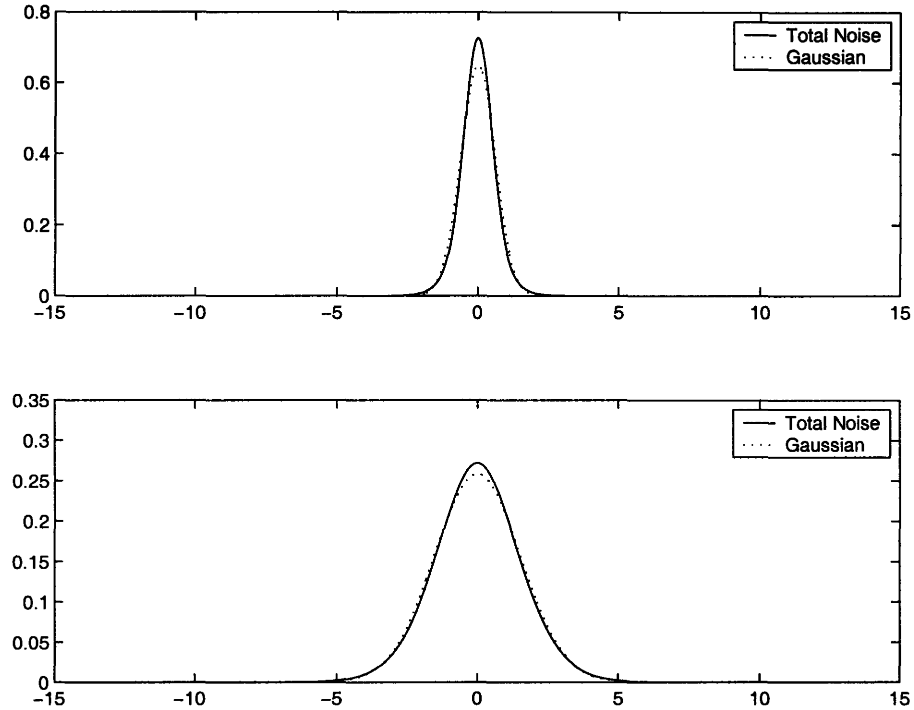


Figure 3-5. Total noise distribution for $C^* = 12$ bits, $L = 24$, and $r = 5$ for symbol with 1 interfering layer (top) and symbol with many interfering layers (bottom) when one sub-channel is off.

is that in [3], using the non-uniform time-varying power allocation, the rates contributed by different layers are equalized. This is not the case in our coding scheme. Here, one sub-block (or equivalently one layer if we remove the staggering) — the one with no interference — dominates in terms of the mutual information contributed at high SNR. This is why increasing the number of layers linearly with C^* is not enough to make this dominating sub-block 'low' SNR and hence repetition incurs a mutual information penalty.

■ 3.3.5 Summary of Results of Efficiency Analysis

From the approximate and exact analyses on the efficiency of the sub-block structured rateless scheme, we can make the following conclusions:

- For an arbitrary rate R^* of the base code, there exists an upper bound on the achievable efficiency. This is found in (3.15) and does not depend on C^*. To achieve a desired efficiency it is necessary to pick the base code rate below the threshold R^* for which (3.15) is tight.

- When increasing L linearly with C^*, the efficiency drops as $1/r$ in the limit of large C^*'s. Hence for a fixed R such that $C^*/L = R$, and a reasonable desired efficiency,

Table 3.5. Efficiencies for $C^*/L = 1/2$ bit with non-uniform power allocation. The power allocation is that of the rateless code in the SISO channel derived in [3].

C^*	1 bit	2 bit	3 bit	6 bit	12 bit
L	2	4	6	12	24
$r = 2$	0.8758	0.8879	0.9032	0.9399	
$r = 2$ Approx	0.8745	0.8875	0.9031	0.9399	0.9691
$r = 3$	0.8344	0.8413	0.8505		
$r = 3$ Approx	0.8338	0.8410	0.8502	0.8771	0.9164
$r = 4$				0.8437	
$r = 4$ Approx	0.8142	0.8187	0.8246	0.8436	0.8774
$r = 5$	0.8030	0.8059			
$r = 5$ Approx	0.8028	0.8058	0.8099	0.8238	0.8516
$r = 6$	0.7954				
$r = 6$ Approx	0.7953	0.7975	0.8004	0.8110	0.8338

there exists an upper limit on C^* beyond which the desired efficiency cannot be achieved for all number of repetitions, r.

- For $R = C^*/L = 1/6$ bit, the efficiency of the coding scheme is above 90% for C^*'s up to 5 bits and for any $r \leq 6$. In practice, C^* usually does not exceed this limit and hence the codeword scheme has a high efficiency up to a reasonable number of repetitions. If we are not interested in rateless, i.e. $r = 1$, the efficiency is still above 90% for C^*'s up to 5 bits.

■ 3.4 Design Formulation

From the results of the efficiency analyses for the sub-block structured scheme, we conclude that for any base code of arbitrary rate, R, there exists an upper limit on C^* in order to achieve a desired efficiency. If we start increasing C^* beyond this limit the desired efficiency could not be achieved. Hence the best we can do for any desired efficiency and a given rate R is to find a maximum C^* for which the desired efficiency is achievable for any number of repetitions, r. In other words, for a base code of given rate R, we should find a maximum C^* and a correspoding maximum number of layers L which result in the desired efficiency no matter what r is. This leads us to the following optimization problem for code design:

$$\max_{C^*,L} \quad C^* \tag{3.26}$$

$$\text{s.t.}$$

$$\eta \leq \frac{RL}{C^*} \tag{3.27}$$

$$R \leq \frac{1}{2L}\sum_{i=1}^{L}\log(1+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\exp(\frac{C^*}{r}+t)-1}}+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\exp(\frac{C^*}{r}-t)-1}})$$

$$\forall r\,,\,\forall|t| \leq \frac{C^*}{r} \tag{3.28}$$

$$C^* \geq 0$$

Here R and η are the code rate and desired efficiency that we wish to design for. Here, η must be picked below the upper bound in (3.15). Note that if we are not interested in rateless, we still use the same optimization problem but replace the condition $\forall r$ with $r = 1$.

Looking at the constraints, we observe that C^* is maximized when both these constraints are tight. From the first constraint in (3.27), maximum C^* occurs when $L = \lceil \eta C^*/R \rceil$. Therefore the second constraint in (3.28) becomes a function of C^* only. Hence the optimization problem reduces to finding the maximum C^* solution to the following system of equations for any r and then taking the worst case C^* among all r's. This must be done for the two extreme cases of SNR pairs which correspond to $t = 0$ and $t = \pm C^*/r$.

$$\begin{cases} L = \lceil \frac{\eta C^*}{R} \rceil \\ R = \frac{1}{2L}\sum_{i=1}^{L}\log(1+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\exp(\frac{C^*}{r}+t)-1}}+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\exp(\frac{C^*}{r}-t)-1}}) \end{cases}$$

It is therefore important to examine the behavior of the rate function for different values of r to be able to make predictions as to where the worst case occurs. We can analyze the asymptotic behavior of the rate function at large r.

Proposition 3.4.1. *For large r, the maximum achievable rate per layer becomes monotonically decreasing and goes to the limit (cf.* (3.8)*):*

$$\frac{1}{2}\log(1+\frac{2C^*}{L})$$

Proof. To see the behavior of the rate function in (3.4), in the limit of large r we make the approximation:

$$
\begin{aligned}
I_l &= \frac{1}{2L}\sum_{i=1}^{L}\log\Big(1+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\exp(\frac{C^*}{r}+t)-1}}+\frac{r/L}{\frac{i-1}{L}+\frac{1}{\exp(\frac{C^*}{r}-t)-1}}\Big)\\
&\approx \frac{1}{2}\log\Big(1+\frac{r}{L}e^{\frac{C^*}{r}}(e^t+e^{-t})-\frac{2r}{L}\Big) \qquad r \text{ large}
\end{aligned}
$$

Now taking the derivative with respect to r we have:

$$
\begin{aligned}
&\frac{\partial}{\partial r}I_l\\
&= \frac{1}{rL}\frac{\left[re^{\frac{C^*}{r}}(e^t+e^{-t})-C^*e^{\frac{C^*}{r}}(e^t+e^{-t})-2r\right]}{1+(re^{\frac{C^*}{r}}(e^t+e^{-t})-2r)/L}
\end{aligned}
$$

Since $e^{\frac{C^*}{r}}(e^t+e^{-t})\geq 2$, the denominator is always positive. Also since $|t|\leq C^*/r$, $e^t+e^{-t}\leq e^{\frac{C^*}{r}}+e^{\frac{-C^*}{r}}$. The limit of the denominator as $r\to\infty$ has already been calculated in (3.12) and is given by $1+(2C^*)/L$. The maximum value the numerator can take is:

$$
\begin{aligned}
(r-C^*)e^{\frac{C^*}{r}}(e^t+e^{-t})-2r &\leq (r-C^*)e^{\frac{C^*}{r}}(e^{\frac{C^*}{r}}+e^{\frac{-C^*}{r}})-2r\\
&= (r-C^*)(e^{\frac{2C^*}{r}}+1)-2r\\
&= r(e^{\frac{2C^*}{r}}-1)-C^*(1+e^{\frac{2C^*}{r}})\\
&= \left(\frac{2C^*}{r}+\frac{1}{2!}(\frac{2C^*}{r})^2+\cdots\right)r-\left(2+\frac{2C^*}{r}+\frac{1}{2!}(\frac{2C^*}{r})^2+\cdots\right)C^*\\
&= \sum_{i=1}^{\infty}\frac{(2C^*)^{i+1}}{(i+1)!r^i}-\sum_{j=1}^{\infty}\Big(\frac{2C^*}{r}\Big)^j\frac{C^*}{j!}\\
&= \sum_{i=1}^{\infty}\frac{C^{*i+1}}{r^i}\left[\frac{2^{i+1}}{(i+1)!}-\frac{2^i}{i!}\right]\\
&= 0+(\frac{8}{6}-\frac{4}{2})\frac{C^{*3}}{r^2}+\cdots\\
&< 0
\end{aligned}
$$

Note that the term $2^{i+1}/(i+1)!-2^i/i!$ is monotonically decreasing with i and since it is 0 at $i=1$, it will be negative afterwards. Hence the numerator in the limit of large r's is negative and goes to zero at $r\to\infty$ and therefore the rate function is monotonically decreasing with r in the limit of large r's. □

As a result, to find the worst case of the rate function we should search in the range of small r's or as $r \to \infty$. The worst case maximum C^* and the corresponding number of layers is somewhere in the range of small r's or at $r \to \infty$ depending on the problem specifications. From (3.14), we must have $L \leq (2C^*)/(\exp(2R) - 1)$ to be able to support a rate of R as $r \to \infty$ and the efficiency we can design for is at most $\eta(R) \leq (2R)/(\exp(2R) - 1)$. The following example makes the procedure more clear.

Example Code Design

Here we design a sub-block structured rateless coding scheme for the following specifications:

$$\begin{cases} R = \frac{1}{6} \text{ bit} \\ \eta = 0.8889 \end{cases}$$

The above efficiency is the maximum we can hope for using a base-code of rate 1/6 bit for which (3.15) is tight. We require the system to have an efficiency as close as possible to the target η for every r. We are interested in maximum C^* for which this is possible. Therefore the optimization problem reduces to finding the C^* solution to the following system of equations for any r and then taking the worst case C^* among all r's. This must be done for the two extreme cases of SNR pairs which correspond to $t = 0$ and $t = \pm C^*/r$.

$$\begin{cases} L = \lceil \frac{\eta C^*}{r} \rceil \\ R = \frac{1}{2L} \sum_{i=1}^{L} \log\left(1 + \frac{r/L}{\frac{i-1}{L} + \frac{1}{\exp(\frac{C^*}{r}+t)-1}} + \frac{r/L}{\frac{i-1}{L} + \frac{1}{\exp(\frac{C^*}{r}-t)-1}}\right) \\ R = \frac{\log(2)}{6} \\ \eta = 0.8889 \end{cases}$$

Solving these equations, the worst case of maximum C^* happens as $r \to \infty$. The solution is:

$$C^* = 5.06 \text{ bits}, \qquad L = 27$$

3.5 Time-varying Behavior

Here we examine in more detail the unknown time-varying behavior of the SNR in the sub-block structured coding scheme and its implications on the base code design. In the sub-block structured coding scheme, SNR varies within a single codeword. As we have seen before, there are L sub-blocks within a single codeword that see different levels of

Table 3.6. SNR variation within a single codeword for the case of $N_1 = N_2$, $C^* = 5.06$ bits, and $L = 27$. (ΔSNR/MAX SNR = 0.4335).

	ΔSNR	Max SNR
$r = 1$	0.0128	0.0294
$r = 2$	0.0255	0.0589
$r = 3$	0.0383	0.0883
$r = 4$	0.0510	0.1177
$r = 5$	0.0638	0.1472
$r = 6$	0.0766	0.1766

Table 3.7. SNR variation within a single codeword for the case when one sub-channel is off, $C^* = 5.06$ bits, and $L = 27$. (ΔSNR/MAX SNR = 0.6814).

	ΔSNR	Max SNR
$r = 1$	0.0560	0.0823
$r = 2$	0.1121	0.1645
$r = 3$	0.1681	0.2468
$r = 4$	0.2242	0.3290
$r = 5$	0.2802	0.4113
$r = 6$	0.3363	0.4935

interference. This variation is known if the SNR pair is fixed. However, the SNR conditions of the two sub-channels are unknown and could take any realization according to (3.16). Effectively, the base codeword sees a time-varying scalar Gaussian channel in which the time variation is unknown because of the uncertainty in the SNR pair realization. This unknown time-varying nature may affect the performance of the base codes designed for the time varying scalar channels. For known time-variation, we can design good codes for a time-varying scalar Gaussian channel. At low SNR, we expect good binary low rate codes for AWGN channels to perform relatively well for the time-varying channel as well.However, here the time-variation is unknown. For an unknown time-variation, we cannot guarantee a good performance for the underlying base codes. The behavior of the SNR within one codeword is shown in Figures 3-6 and 3-7 for $C^* = 5.06$ bits and $L = 27$ in the two extreme case of SNR pairs. Tables 3.6 and 3.7 show the maximum difference in the SNR values for different r's and also the corresponding highest SNR for each r (SNR of sub-block with no interference).

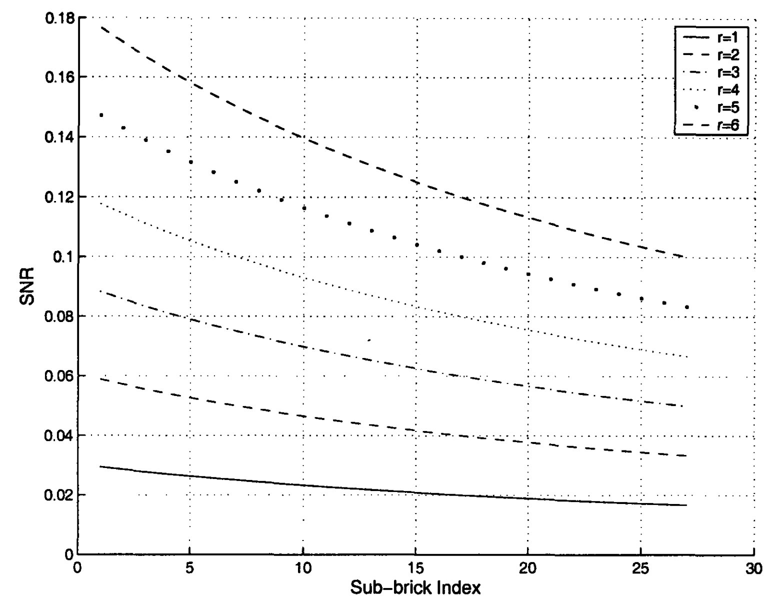


Figure 3-6. SNR variation within a single codeword. Here $C^* = 5.06$ bits , $L = 27$, and $N_1 = N_2$. The maximum variation is 43% of the maximum SNR.

3.6 Going to Higher Maximum Rates: Another Sub-block Structured Code Incorporating the Faster Than Nyquist Signaling

Here we examine another sub-block structured code which builds on the sub-block structure already developed but also incorporates the Faster Than Nyquist signaling (FTN) in its structure. As discussed before, the sub-block structured coding scheme is limited not to go above a maximum C^* in order to have a certain efficiency. For example to have an efficiency of 88% using a base code of rate 1/6 bit, we cannot go above a C^* of 5 bits (per real dimension). We may be dealing with situations where C^* is larger than this number even though in practice this does not usually happen. This implies that using this sub-block structure alone is not enough. In other words, we should somehow introduce a noise floor on the codewords to make the SNR seen by the layers smaller and take the effective C^* to the maximum C^* allowed for the desired efficiency. One approach to this is to use FTN signaling. This idea is used in [3] to design rateless codes for Gaussian single input single output (SISO) channels. We use MMSE-DFE decoding, where decisions and subsequent interference cancellation is done on coded blocks which according to Guess-Varanasi in [4] is capacity approaching. For complete analysis refer to [3] and [4]. This section presents the basic ideas of applying FTN to construct another sub-block structured code. More careful analysis of this scheme and its performance will be a valuable future direction to take.

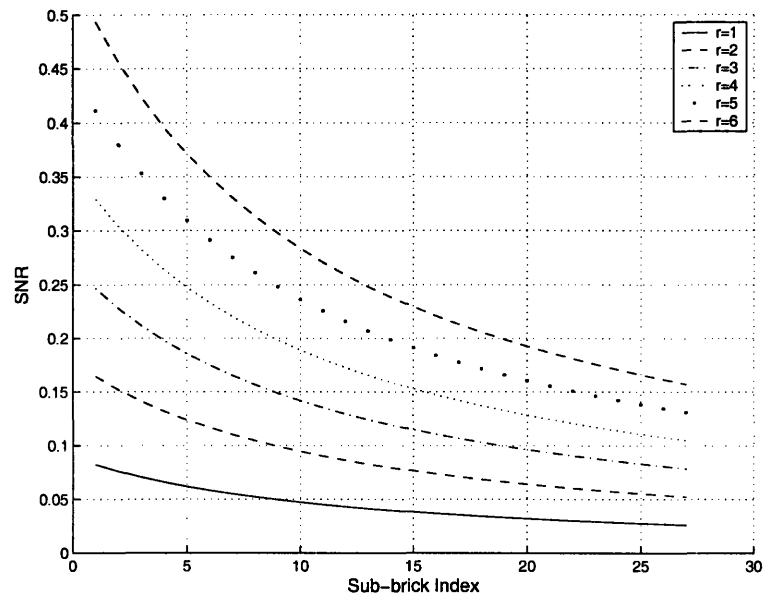


Figure 3-7. SNR variation within a single codeword. Here $C^* = 5.06$ bits, $L = 27$, and one sub-channel is off. The maximum variation is 68% of the maximum SNR.

To implement the FTN scheme, we first form the code block in the previous sub-block structure as in Figure 3-8. Here the superposition code, which is the sum of all layers, is shown. For simplicity we have separated this superposition code into blocks of length n, where n is the base code block length, and denoted these blocks $\mathbf{x}_1, \cdots, \mathbf{x}_M$. Note that this separation is just to make the choice of notations simple. In reality the codewords are staggered and hence any length n block of this form is constructed by adding different number of sub-blocks of the layer codewords. To implement FTN signaling, we form and send the interleaved sequence:

$$\underbrace{0, \cdots, 0}_{K}, \mathbf{x}_1(1), \cdots, \mathbf{x}_M(1), \underbrace{0, \cdots, 0}_{K}, \mathbf{x}_1(2), \cdots, \mathbf{x}_M(2), \cdots, \underbrace{0, \cdots, 0}_{K}, \mathbf{x}_1(n), \cdots, \mathbf{x}_M(n) \quad (3.29)$$

Here the length of ISI is assumed to be K. To make the efficiency loss due to sending K zeros negligible, we need M units of the layered code where $M \gg K$. Note from (3.3) that we already had a condition on M to be $M \gg 1$.

To find out how much faster than Nyquist we should send the symbols, we use the fact that C^* should be taken to maximum allowed C^* for a certain efficiency. Let's call this maximum, $C^*_{\max}(\eta, R)$. From [3] using FTN with MMSE-DFE decoding results in an effective SNR of $\mathrm{SNR}_{\text{MMSE-DFE-U}} = (1 + P/(WN_0))^{T/T_{\text{Nyq}}} - 1 = (1 + \mathrm{SNR})^{T/T_{\text{Nyq}}} - 1$, where

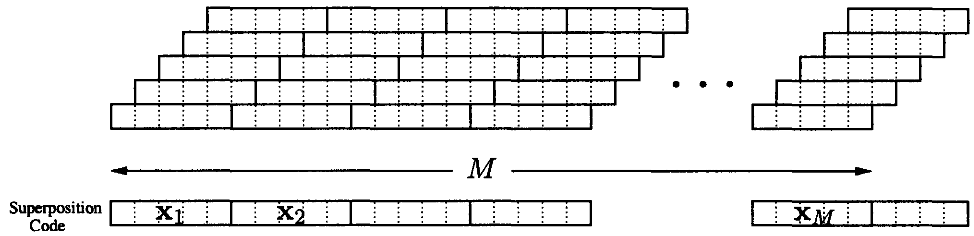


Figure 3-8. The sub-block structured code is first constructed and the superposition code is formed. To implement the FTN, the interleaved sequence, $0, \cdots, 0$, $\mathbf{x}_1(1), \cdots, \mathbf{x}_M(1)$, $0, \cdots, 0$, $\mathbf{x}_1(2), \cdots, \mathbf{x}_M(2)$, $\cdots, 0, \cdots, 0$, $\mathbf{x}_1(n), \cdots, \mathbf{x}_M(n)$ is then sent faster than Nyquist over the sub-channels. It is assumed that the length of ISI is K. That is why the number of zeros sent every time is K.

W is the bandwidth, N_0 is the one-sided noise power spectral density, T_{Nyq} is the Nyquist sampling period, and T is the sampling period we should design for. Here SNR per symbol in the channel is $\text{SNR} = P/(WN_0)$. Let us define $\gamma = T_{\text{Nyq}}/T$ which is the over-sampling ratio. We want to have:

$$\frac{1}{2}\log(1 + \text{SNR}_{1\text{-MMSE-DFE-U}}) + \frac{1}{2}\log(1 + \text{SNR}_{2\text{-MMSE-DFE-U}}) = C^*_{\max}(\eta, R)$$

Using the expression for SNR-MMSE-DFE-U we have:

$$\begin{aligned} C^*_{\max}(\eta, R) &= \frac{1}{2}\log(1 + (1 + \text{SNR}_1)^{\frac{1}{\gamma}} - 1) + \frac{1}{2}\log(1 + (1 + \text{SNR}_2)^{\frac{1}{\gamma}} - 1) \\ &= \frac{1}{2\gamma}\log(1 + \text{SNR}_1) + \frac{1}{2\gamma}\log(1 + \text{SNR}_2) \\ &= \frac{C^*}{\gamma} \end{aligned}$$

We should therefore pick $\gamma = C^*/C^*_{\max}(\eta, R)$.

Hence, the coding structure is as follows: we repeat dithered versions of the block-structured code consisting of M units over the two sub-channels, interleave the block on each sub-channel according to (3.29), and send the symbols faster than Nyquist over the two sub-channels with an oversampling rate of $\gamma = C^*/C^*_{\max}(\eta, R)$. We repeat the same procedure for dithered copies of the sub-block structured code in time (if rateless). The next section explains this in more detail.

3.6.1 Encoding and Repetition Scheme

We have to set a target efficiency and a target rate R for our scheme, where R is in the range of rates for which this efficiency is achievable. We find the maximum allowed C^* and therefore L for this efficiency by solving the optimization problem in (3.26) and use those for the rest of our design. We choose a good base codebook of the desired rate R, pick each layer independently from this codebook, dither them, and form the basic unit of the sub-block structured code as in Figure 3-1. We do this M times, i.e., pick ML codewords and combine them in the overall block and form the superposition code in Figure 3-8. We repeat a dithered version of the same superposition block on the second sub-channel and in time (if rateless) as well. We then interleave each copy of the overall block as in (3.29) and send all the repeated interleaved structures faster than Nyquist with a signaling rate of $T = T_{\text{Nyq}}/\gamma$ ($\gamma = C^*/C^*_{\max}(\eta, R)$) one after the other.

3.6.2 Decoding

At the decoder we use the block DFE structure shown in Figure 3-9 (see [3]). A user with $C(r) = C^*/r$ collects r copies of the overall interleaved structure, applies FFE filtering to each structure, deinterleaves the symbols to reconstruct the overall code blocks (Figure 3-8), and then applies MRC to the copies of each block, i.e., averages them. It then applies the MMSE-DFE on the combined structure after the MRC. The decoder in the loop here is the successive decoder for the original sub-block structured scheme which decodes the layers in the basic units. The first unit in the combined structure (see Figure 3-8) sees only precursor ISI. The decoder in the loop first decodes the layers of the first unit. It then subtracts its effect from the second unit and then decodes the layers of the second unit and continues with this successive decoding for all M units. (Note we have neglected the edge effects between the copied structures). Each of the M units at the end contributes to a rate of C^*/γ. Therefore γ of these units will give us an effective rate of C^*.

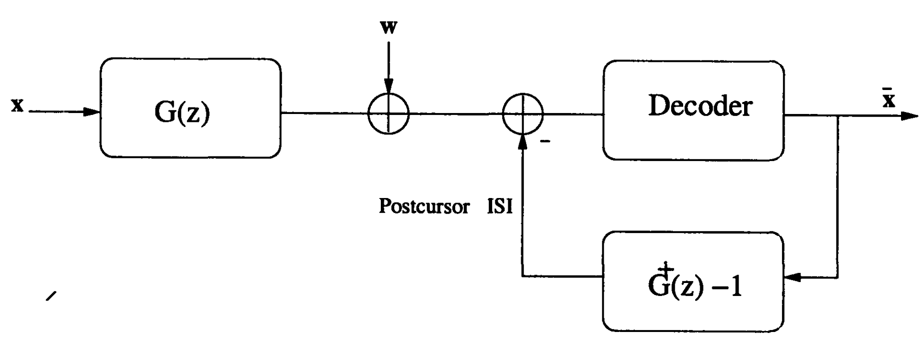


Figure 3-9. MMSE-DFE structure used. Here $G(z) = H(z)\text{FFE}(z)$ and $W(z) = \text{FFE}(z)Z(z)$, where $H(z)$ is the Z-transform of the channel response and $Z(z)$ is the Gaussian noise. $G^+(z)$ is the Z-transform of the causal part of g.

Comparison of the Combined Block-structure and FTN Scheme with FTN Alone

In [3], FTN is used as an alternative to the time-varying non-uniform power allocation layering scheme. The layer codewords are interleaved and sent faster than Nyquist. The ISI introduced on the layers using FTN takes the effective SNR of each layer to a low SNR regime for which repetition is efficient. Basically, the FTN signaling alone generates the layering effect.

We now discuss the advantages of the combined block-structure and FTN scheme over using FTN alone. In the FTN scheme all the layering comes from ISI and therefore we must send much faster than Nyquist to take the layer SNR's to the low SNR regime. In the combined scheme proposed above, only part of the layering comes from ISI. Basically we just need to introduce enough ISI to take the effective C^* to the maximum C^* allowed for a desired efficiency and rate per layer. Hence we do not need rates much faster than Nyquist and therefore much less interference is introduced which will probably improve the performance of the DFE. For example if we send twice as fast as Nyquist, only every other symbol (half of the other symbols) will generate interference with any one symbol. Sending twice as fast as Nyquist and using a rate of $R = 1/6$ bits, allows us to handle C^*'s as large as 10 bits (per real dimension) with an efficiency of almost 88% if we assume no performance loss due to FTN.

Hence we have a basic idea how to incorporate FTN into the block-structured scheme. However, the analysis is not complete and needs careful study in terms of its efficiency performance. Again as we mentioned before, the sub-block scheme without FTN is fairly efficient for the C^*'s that usually arise in practice. Hence in practice, there may be no need to use FTN.

■ 3.7 Summary

Here we summarize the highlights of this chapter:

- The sub-block structured coding scheme is capacity achieving in the limit of $L \to \infty$, rateless or not.

- For the scheme to be approximately capacity achieving, L must increase exponentially with C^*.

- Increasing L linearly with C^* results in reasonable high efficiencies for practical ranges of C^*. For example for $C^*/L = 1/6$ bit, the efficiency of the scheme is above 90% for C^* values up to 5 bits (per real dimension) and up to $r = 6$ repetitions.

- For any fixed rate base-code and a valid desired efficiency, there is a maximum C^* above which the efficiency cannot be achieved.

- If designing for C^* values higher than the maximum allowed value for a certain efficiency, we could incorporate FTN to take the effective C^* to the maximum allowed one and still achieve the desired efficiency.

- The disadvantage of the sub-block structure is the unknown time-varying SNR behavior within a single codeword. As a result of this unknown time-varying nature, good base codes for a known time-varying scalar Gaussian channel are not guaranteed to perform close to capacity. More specifically, at low SNR, using good low rate AWGN base codes does not guarantee to perform close to capacity.

Chapter 4

Conclusions

This thesis has been motivated by the need to design low-complexity capacity approaching universal and rateless codes for parallel Gaussian channels. Low-complexity rateless capacity approaching codes have been designed for SISO channels in [3] but little has been done for the parallel channel problem. Approximately universal codes have been proposed for parallel Gaussian channel in [8] and [9]. These codes, however, do not make use of simple scalar Gaussian channel base codes. In our design architecture, we will convert the parallel Gaussian channel into a set of scalar Gaussian channels and hence use low-complexity 'good' base codes for the corresponding scalar channel to communicate. Basically in our design, the code effectively sees a scalar Gaussian channel. Hence our architecture is different from that of [8], [9].

In Chapter 2, we study the design of layered universal codes for parallel Gaussian channels with deterministic dither which employ a standard AWGN base code. In this chapter we assume to have a fixed maximum rate, C^*, and require the code to be capacity achieving regardless of the relative quality of the two sub-channels. Even though the design can be extended to be rateless for situations where the maximum rate is not fixed and is changing as well, we only focus on the universality of the design in terms of the relative quality of the two sub-channels with a fixed overall maximum rate for compactness of exposition. Considering only this uncertainty is motivated by the outage definition in slow fading channels. The coding scheme repeats the same code on the two sub-channels with possibly different dithers. The decoder uses an MMSE receiver along with successive cancellation. The main strategies used in the design are layering, deterministic optimal dithering, and grouping and combining of layer codeword symbols using unitary transformations. We study the efficiency performance of these codes and investigate the effect of the number of layers and the dither dimension on this performance. We show that increasing the dither dimension does not improve the efficiency by any noticeable amount and real ± 1 dithers perform just as well. The efficiency increases by adding layers up to a certain L and after that it saturates. We derive an expression for this saturation efficiency as a function of the maximum rate of the channel. Using only layering, grouping, and dithering, the highest performance gain over the simple case of random dithering and repeating over the two sub-channels happens at a $C^* = 3.76$ bits and is 13.75%. We then consider improvements to the scheme by examining the effect of 1 bit of CSIT and partial channel information on the performance. We show that even a single bit of CSIT significantly improves the efficiency of the scheme. We then examine the robustness of the scheme to unreliable channel information and show that the coding is very robust to this unreliability. Finally we study the effect of the decoder structure on the performance and show that in the simple case with no CSIT, an MRC receiver performs close to an MMSE receiver. However, the MMSE receiver achieves the saturation efficiency at a small number of layers whereas for the MRC this is achieved at an infinite number of layers.

In Chapter 3, we focus on another universal code and extend it to be rateless as well. We design a sub-block structured code that uses layering, random Bernoulli (1/2) dithering, and staggering of the layer codewords. The code is again repeated over the two sub-channels with independent dithers. The decoder employs an MRC receiver along with successive cancellation. Staggering of layers combined with the order of decoding and successive cancellation, makes the coding scheme symmetric with respect to all layers. For the rateless version, the code is also repeated in time with independent dithers. We show that this code is capacity achieving in the limit of a large number of layers as long as the number of layers is increased exponentially with C^*. Increasing the number of layers exponentially with C^* is not practical and does not allow a single base code to be used for the scheme. Thus, we examine the efficiency of the scheme when L is increased linearly with C^* and perform approximate and exact efficiency analyses. From the approximate efficiency analysis, we derive lower and upper bounds on the efficiency and show that they can be made tight. The exact efficiency analysis is performed for intermediate values of C^* and shows that for practical ranges of C^* (up to 5 bits per real dimension) the scheme has reasonably high efficiency of around 90% using a ratio $C^*/L = 1/6$ bit. We then formulate a design problem where for any base code of a certain rate R^* and a valid efficiency below the upper bound for that rate, we find the maximum C^* for which this efficiency is achievable.The draw back of this sub-block structured scheme is its unknown time-varying SNR behavior in a single codeword which may affect the performance of the underlying base codes. We then briefly examine the use of FTN signaling which will take the effective C^* of the channel to the maximum allowed C^* for a certain efficiency and base code rate and hence allow us to go to higher C^* values with high efficiency.

There are still some aspects of the two coding schemes that need further analysis. The effect of the decoder structure in the layered coding scheme of Chapter 2 needs further study. For example the effect of the decoder structure for the case when 1 bit of CSIT is available is interesting to study. The use of FTN signaling in the sub-block structured code of Chapter 3 needs to be analyzed further in terms of the efficiency loss due to using faster than Nyquist signaling and the filters involved. Also, the effect of the unknown time-varying SNR behavior in the sub-block structured code on the performance of the base codes, especially on the performance of good AWGN codes at low SNR, needs to be studied and simulated.

Bibliography

[1] T. M. Cover and J. A. Thomas. *Elements of Information Theory.* Wiley, New York, 1991.

[2] P. Dita. Factorization of unitary matrices. *Journal of Physics A: Mathematical and General*, 36(11), March 2003.

[3] U. Erez, G. W. Wornell, and M. D. Trott. Faster-than-nyquist coding: the merits of a regime change. In *42nd annual Allerton conference on communications, control, and computing, Monticello, Illinois*, October 2004.

[4] T. Guess and M. K. Varanasi. An information-theoretic framework for deriving canonical decision-feedback receivers in Gaussian channels. *IEEE Trans. Inform. Theory.* (to appear).

[5] M. Luby. LT-codes. In *Proceedings of the ACM symposium on Foundations of Computer Science (FOCS)*, 2002.

[6] A. Shokrollahi. Raptor codes. (preprint, 2002).

[7] John R. Silvester. Determinants of block matrices. *Math Gazette 84*, 2000.

[8] S. Tavildar and P. Viswanath. Approximately universal codes over slow fading channels. *IEEE Trans. Inform. Theory*, 2006.

[9] D. Tse and P. Viswanath. *Fundamentals of Wireless Communication.* Cambridge University Press, Cambridge, 2005.